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On the First Integrals and Linearization of Force-Free Duffing Van der Pol Equation

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ABSTRACT

The linearization problem of second-order nonlinear differential equations was first explored by Sophus Lie. In line with this research, he discovered that these equations must satisfy certain conditions to be linearized, and examined the structures of linearization transformations performed to fulfil these conditions. In this study, one of the transformation methods is performed by these conditions is discussed. It is shown that the linearized equations can be integrated by obtaining the first integrals. Then, the force-free Duffing-van der Pol oscillator is examined. The necessary conditions on parameters for the linearization of this equation are obtained, and the first integral of this equation was found.

1. Introduction

The first integrals and solutions of oscillator equations are receiving significant attention in the literature since these equations are essential in applied mathematics, physics, and engineering problems. Obtaining the analytical solutions to the oscillator problem is more complicated than the numerical solutions. Some of the feasible methods used to solve nonlinear oscillator problems can be presented as the linearization method [1], nonlocal transformation [2], the Hamiltonian method [3], parameter-expansion method [4], max-min method [5], and the extended direct algebraic method [6].

One of these valuable methods is symmetry: an important method for investigating nonlinear equations. Therefore, many researchers have been interested in symmetry methods [7-13]. There are different methods to find symmetries; one of these methods is to get the first integrals. Obtaining solutions to linear differential equations is more effortless by comparing nonlinear equations. Moreover, finding solutions for nonlinear differential equations is a rather complex task. In contrast, obtaining approximate solutions of nonlinear equations is straightforward, while it is known that getting analytical solutions is a complicated procedure.

One method of obtaining solutions to nonlinear equations is to turn the equation into a known linear equation called linearization. In this study, linearization methods are examined for the nonlinear equations of the form

$$\ddot{x}+a_3(t,x)\,\dot{x}^3+a_2(t,x)\,\dot{x}^2+a_1(t,x)\,\dot{x}+a_0(t,x)=0, \quad (1)$$

which t is an independent variable and x is a dependent variable of t [1].

Equations in form (1) have first integrals of form

$$A(t,x)\dot{x}+B(t,x). \quad (2)$$

It is known to have the first integrals, and a feasible procedure has been developed to calculate the first integrals [14]. Using this procedure to find the first integrals of form (2), the coefficient a_3 in equation (1) should be zero. Depending on this condition, equation (1) is converted to

$$\ddot{x}+a_2(t,x)\,\dot{x}^2+a_1(t,x)\,\dot{x}+a_0(t,x)=0. \quad (3)$$

Using this feasible procedure, we examine how to find the first integrals and integrating factors of nonlinear equations. Then, we apply this procedure to our equation and find these functions.

2. Materials and Methods

The Method for Constructing the First Integrals and Integrating Factors

We investigate the first integrals in form A(t,x) ẋ+B(t,x) and how equations in form (3) are linearized using these first integrals; and to perform it, we classify equations to obtain the first integrals in this form. To classify the equations, these functions are defined as

$$S_1(t,x)=a_{1x}-2a_{2t}, \quad (4)$$

$$S_2(t,x)=(a_0a_{2}+a_{0x})_x+(a_{2t}-a_{1x})_t+(a_{2t}-a_{1x})a_1. \quad (5)$$

After these definitions, the function $S_1(t,x)$ is computed; if $S_1=0$, then the function S_2 should be zero. If the function is

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$S_1 \neq 0$, then two different functions should be used. These can be given

$$S_3(t,x) = \left(\frac{S_2}{S_1}\right)_x - (a_{2t-}a_{1x}), \tag{6}$$

$$S_4(t,x) = \left(\frac{S_2}{S_1}\right)_t + \left(\frac{S_2}{S_1}\right)^2 + a_1 \left(\frac{S_2}{S_1}\right) + a_0 a_{2+} a_{0x}. \tag{7}$$

If the function S_3 is computed as zero for these two new functions, then it is seen that $S_4 = 0$. Two different linearizing procedures are used according to this classification, and the appropriate procedure is chosen for the considered equation with respect to obtaining classification results. We investigate the following propositions to explain these procedures that give first integrals [14].

Theorem 1: Suppose that the equation is in form (3) and the functions S_1 and S_2 are calculated. In this situation, two different cases and procedures are presented:

Case I: We suppose that $S_1 = 0$ and $S_2 = 0$ in this case.

Our first aim in these procedures is to yield the function P and to do it, the derivatives of P are

$$P_t = \frac{1}{2}a_1, \quad \text{and} \quad P_x = a_2 . \tag{8}$$

Defining the function $f(t,x)$ as

$$f(t,x) = a_0 a_{2+}\, a_{0x} - \frac{1}{2}a_{1x} - \frac{1}{4}a_1{}^2, \tag{9}$$

this function is derived. After that, the function $f(t)$ is substituted in

$$g''(t) + f(t)g(t) = 0, \tag{10}$$

and solving equation (10), the function $g(t)$ is found.

Later, we should find the function $Q(t,x)$ and to do it, the derivatives of $Q(t,x)$ are written as

$$Q_{t=}a_0\, g\, e^P \quad \text{and} \quad Q_{x=}\left(\frac{1}{2}\, a_1 - \frac{g'}{g}\right) g e^P. \tag{11}$$

And solving equation (11), Q is found.

Thus, the coefficients of A and B are computed such that

$$A = g\, e^P \quad \text{and} \quad B = Q. \tag{12}$$

Hence, the first integrals of the equation are yielded.

Case II: $S_1 \neq 0$ and S_3 and S_4 should be zero.

For this case, the function P is obtained using the following equations

$$P_t = a_1 + \frac{S_2}{S_1} \quad \text{and} \quad P_x = a_2 . \tag{13}$$

The derivatives of Q are given

$$Q_{t=}a_0\, e^P \quad \text{and} \quad Q_{x=-}\left(\frac{S_2}{S_1}\right) e^P . \tag{14}$$

The coefficients of the first integral are defined

$$A = e^P \quad \text{and} \quad B = Q. \tag{15}$$

Theorem 2: We take equation in the form (3), and it has the first integral in the form I = A(t, x) $\dot{x}$ + B(t, x). Additionally, it is known that this equation has an integrating factor that forms

μ =A(t, x).

3. Results and Discussion

First integrals and Integrating Factors of force-free Duffing-van der Pol Equation

In this section, we consider the force-free Duffing-van der Pol oscillator equation with nonlinear damping

$$\ddot{x} + (\alpha + \beta\, x^2)\dot{x} - \gamma\, x + x^3 = 0, \tag{16}$$

where x is the position coordinate, t time, and γ is a scalar parameter demonstrating the damping's nonlinearity and strength [15]. Equation (16) is an autonomous equation expressing dispersion of voltage pulses along a neuronal axon.

Balthasar van der Pol discovered the equation (16). It yielded stable oscillations, renamed relaxation oscillations, and its current name is a limit cycle type in electrical circuits using vacuum tubes. This equation is very famous in many areas such as physics, biology, sociology, and even economics, because this equation has not only physical meaning but also biological meaning. Therefore, it is used to model electrical circuits on the one hand, and to measure the electrical potentials of neurons in the stomachs of living things on the other hand [16].

Moreover, this equation was used to model the action potentials of neurons by Fitzhugh and Nagumo [17]. Additionally, it is used in seismology as a model of the two plates in a geological fault and phonation as a model for the right and left vocal fold oscillations. Thus, earthquake faults with viscous friction can be characterized by this equation.

The analytical solutions of the oscillator equations with nonlinear damping are still not explored by researchers; therefore, articles are mostly interested in damped free oscillator equations [18-19]. In addition to this, Panayotounakos and his collaborates coworkers demonstrated that this equation is not analytical without linear stiffness terms [20]. Therefore, researchers have investigated for approximate solutions to this equation using numerical methods. The approximate solutions of this equation are obtained by a new homotopy perturbation method, the Runge-Kutta method [21], and the differential transform method [22]. Then, Chandrasekar et al. [23] examined the first integrals and exact solutions of this equation with special choices for $\alpha = 4/\,\beta$ and $\alpha = -3/\beta^2$.

As can be seen from the studies performed on this equation in the literature, since the solutions of this equation could not be found, its numerical solutions were investigated. In the study conducted in [23], the first integral was found for some special cases. First integrals and integrating factors for the general form of this equation have not been found before previously. The first integral for the general form of this equation was first obtained in this article.

We now apply the procedure examined in the previous section to find the first integral of Duffing van-der Pol equation. It is known that we should classify this equation according to given functions to apply the procedure and, thus we classify equation (16) computing the functions S_1, S_2, S_3 and S_4. First, $S_1 = 2\alpha x$ is found, and it is shown that the function is $S_1 \neq 0$. Hence, the functions S_3 and S_4 should be equal to zero. We calculate these functions to complete classification and find $S_3 = 0$ and the function S_4 as

$$S_4 = \frac{9-3\alpha\beta-\beta^2\gamma}{\beta^2}\ . \tag{17}$$

Since S_4 should be equal to zero, we suppose that

$$9 - 3\alpha\beta - \beta^2\gamma = 0\ . \tag{18}$$

And from (18), the parameter γ is yielded

$$\gamma = -\frac{3(-3+\alpha\beta)}{\beta^2}\ . \tag{19}$$

Along these lines of the results, we can say that equation (16) can be classified, and it is in the second class. Hence, case II should be applied to this equation to obtain its first integrals and integrating factors. To get them, we first find the function P as

$$P(t) = \frac{3\,t}{\beta}. \tag{20}$$

We know the following equations

$$Q_{t=} e^{\frac{3\,t}{\beta}}\left(x^3 + \frac{3x(\alpha\beta-3)}{2x\beta}\right), \tag{21}$$

and

$$Q_x = -\frac{e^{\frac{3\,t}{\beta}}\left(6x - 2x\,\beta\,(\alpha+\beta\,x^2)\right)}{2x\beta}. \tag{22}$$

If we solve the equations (21) and (22), $Q(t,x)$ is found below

$$Q(t,x) = \frac{e^{\frac{3\,t}{\beta}}\left(-3x + x\,\alpha\,\beta + \frac{x^3\,\beta^2}{3}\right)}{\beta} + c_3. \tag{23}$$

Thus, the functions of A and B

$$A(t,x) = e^{\frac{3\,t}{\beta}}, \tag{24}$$

and

$$B(t,x) = \frac{e^{\frac{3\,t}{\beta}}\left(-3x + x\,\alpha\,\beta + \frac{x^3\,\beta^2}{3}\right)}{\beta} + c_{3,} \tag{25}$$

are found. Finally, the first integral of equation (16)

$$I = e^{\frac{3\,t}{\beta}}\,\dot{x} + \frac{e^{\frac{3\,t}{\beta}}\left(-3x + x\,\alpha\,\beta + \frac{x^3\,\beta^2}{3}\right)}{\beta} + c_{3,} \tag{26}$$

is yielded. And integrating factor of the equation (16) is derived as

$$\mu = e^{\frac{3\,t}{\beta}}\ . \tag{27}$$

It is seen that this first integral and integrating factor include arbitrary parameters; therefore, different first integrals can be found with other choices for these arbitrary parameters, and a new classification can be made here. If the special conditions taken in the study in [23] are substituted in the first integral (26), we found that the first integral is obtained for special cases in mentioned study.

4. Conclusion

In this study, t first integrals and integrating factors of force-free Duffing van-der Pol equation are investigated by some methods. Firstly, this equation is classified, and the linearization procedure is applied, then, its first integral and integrating factor, including arbitrary parameters, are found. After that, these first integral and integrating factors can be classified for different choices of arbitrary parameters.

It is known that numerical solutions of this equation were investigated in the literature since the analytical solutions of this equation could not be found, and then the first integral of this equation was found for some special cases by some researches. Hovewer, first integrals and integrating factors of the general form of this equation have not been found previously. The first integral and integrating factor of the general form of this equation were first obtained in this study.

Author Contribution

Conceive-O.Orhan; Design-O.Orhan; Supervision-O.Orhan; Experimental Performance, Data Collection and/or Processing-O.Orhan; Analysis and/or Interpretation-O.Orhan; Literature Review-O.Orhan; Writer-O.Orhan; Critical Reviews-O.Orhan

Conflict of interests

The author has declared no conflicts of interest.

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