Explicit algebraic classification of vacuum, shearfree and non-twisting spacetimes at Large $D$

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**Research Article**

**ABSTRACT**

Most general, shearfree and twistfree geometry is revisited and its Weyl scalars are obtained with large $D$ limit. It is shown that, the spacetime is Type I(b) or more special in this limit like the classification of any arbitrary dimension $D > 4$. As an example, classification of vacuum RT spacetime is investigated. As expected, the spacetime becomes algebraically special and it is Type II or more special. Obligatory conditions are determined for other types and sub-types as the dimension of the spacetime $D \rightarrow \infty$.

1. Introduction

After Robinson-Trautman (RT) solution in 4-dimensions [1, 2] was obtained, the results which enable to understand blackhole physics, theory of gravitational waves and cosmology were commonly studied. As the spacetime geometrically defines a shear-free, twist-free and expanding congruence of null geodesics, it involves many well-known solutions i.e. Schwarzschild, Reissner-Nordström, Schwarzschild-de Sitter, Vaidya, C-metric. So, the RT spacetime in 4 dimensions has include various algebraically special Petrov-Penrose types which are analyzed in [3, 4].

RT solutions in empty space generalized to higher dimensions in [5], but surprisingly higher dimensional results does not include as many as solutions like $D = 4$. Additionally, aligned electromagnetic fields within Einstein-Maxwell theory [6] and general $p$-form Maxwell fields [7] were associated with higher dimensional RT spacetime to analyze richness of it. After a classification scheme for the Weyl tensor of higher dimensional spaces with Lorentzian signature was put forward [8, 9] (developments and applications of the classification of the Weyl tensor in higher dimensional Lorentzian geometries is reviewed in [10]), classification of higher dimensional RT spacetime was explicitly analyzed [11].

On the other hand, Emparan and et al improved a new perspective to higher dimensional solutions with large $D$ expansion method [12–22]. More concretely, the limit $D \rightarrow \infty$ results in a convenient simplification of the equations and possibly also a novel reformulation of the dynamics [23]. Although algebraic classification of RT is studied [24] as the dimension of the spacetime $D \rightarrow \infty$, algebraically special Petrov types and subtypes of vacuum RT spacetime are discussed for the first time.

The paper is organized as; Section 2 we revisit higher dimensional shear-free, non-twisting and expanding metric and its Weyl scalars. By defining boost weight and vanishing Weyl scalars the algebraic classification of the general metric is summarized. Main purpose of the paper is studied in Section 3. Algebraic classification of the vacuum RT spacetime for primary and secondary WANDs are investigated with obligatory conditions.

2. Algebraic structure of the Weyl tensor at large $D$

$D$ dimensional most general, shear-free, twist-free metric can be written in the form [5]:

$$ds^2 = g_{pq} (u, r, x) dx^p dx^q + 2g_{up} (u, r, x) du dx^p - 2udu^2 + g_{uu} (u, r, x) du^2$$

where latin indices $p, q, ...$ count to 2 to $(D - 1)$ and $x$ is shorthand of these $D - 1$ spatial coordinates on the traverse space. Non-twisting structure of the spacetime causes a null foliation by null hypersurfaces $u =$const. which ensures to define the coordinate $u$. Equivalently, a non-twisting null vector field $k$ that is everywhere tangent (and normal) to $u =$const. can be defined. So, the affine parameter $r$ along a null geodesic congruence generated by $k$ is determined as the second coordinate which gives $k = \partial_r$. The relations between covariant and contravariant metric components of the metric 1 become:

$$q^{\alpha\beta} = -1, q^{\alpha p} = g^{pq} g_{uu}, g^{rr} = -g_{uu} + g^{pq} g_{qp} g_{uu}. \text{ In addition, the traverse space metric can be introduced as } g_{pq} = R^2 (u, r, x) h_{pq} (u, x) \text{ where } R = \exp \left( \int \Theta (u, r, x) dx \right) \text{ and } \Theta \text{ is corresponding to the expansion. Although } \Theta = 0 \text{ (non-expanding case) corresponding Kundt spacetime, we will analyze expanding case that is RT spacetime.}$$

The most natural null frames for the metric (1) can be written in the form:

$$k = \partial_r, \quad \ell = \frac{1}{2} g_{uu} \partial_r + \partial_u, \quad m_i = m_i^p (g_{up} \partial_r + \partial_p),$$

where they satisfy the normalization conditions; $k, \ell$ are normalized as $k, \ell \rightarrow \lambda k, \ell \rightarrow \lambda^{-1} \ell$ and $m_i \rightarrow m_i, \ell$; boosts are obtained. One summarizes the boost weight of the null basis $+1, -1, 0$, respectively [25].
Weyl scalar components of the metric (1) for the null frame can be obtained with large $D$ expansion as:

$$\Psi_{ij} = C_{abcd} \delta^{a b} \delta^{c d} m_i^m m_j^m C_{prq} = 0,$$

$$\Psi_{TT} = C_{abcd} \delta^{a b} \delta^{c d} m_i^p m_j^p C_{urp} = m_i^p m_j^p C_{urp},$$

$$= m_i^p \left[ \left( -\frac{1}{2} g_{u p, r} + \Theta g_{u p} \right) , + \Theta p \right],$$

$$\Psi_{ijk} = C_{abcd} m_i^m m_j^m m_k^m = m_i^m m_j^m m_k^m C_{prm} = 0,$$

$$\Psi_{2G} = C_{abcd} \delta^{a b} \delta^{c d} = C_{ur},$$

$$= \left( \frac{1}{2} g_{u u, r} - \Theta g_{u u} \right) , - \frac{1}{2} g_{u u, r} g_{u u, r}, - 2 \Theta, u,$$

$$+ \Theta g_{u u, r}, - \Theta g^{u r} g_{u r}.$$

$$\Psi_{2T/ij} = C_{abcd} m_i^m m_j^m m_j^m = m_i^m m_j^m m_j^m C_{rqq} + g_{u r} C_{ur},$$

$$+ \frac{1}{2} g_{u u} C_{prq},$$

$$m_i^m m_j^m \left[ \left( g_{u p, u u, r} + \frac{1}{2} g_{u r, u r}, + \frac{1}{2} g_{u u} g_{u r, u r} \right) \right],$$

$$+ \frac{1}{2} g_{u u} \left( g^{u r}, u r \right),$$

$$+ g_{u p} g_{u u, r} + \Theta \left( 2 g_{u p} g_{u r, r} + E_{p m} - E_{m p} \right),$$

$$\Psi_{2T} = C_{abcd} \delta^{a b} \delta^{c d} m_i^m = m_i^m \left( \frac{1}{2} g_{u u} C_{ur}, \right),$$

$$+ g_{u p} C_{ur} + \Theta = m_i^m \left( \frac{1}{2} g_{u u} C_{ur} \right),$$

$$- g_{u u, p} + \Theta, u, + \frac{1}{2} g_{u u} \Theta, p - \frac{1}{2} g_{u u} g_{u p}, \Theta, r,$$

$$+ \frac{1}{2} g^{u m} g_{u m} C_{r u}, - \Theta g_{u r, u} - \Theta g_{u u}, - \Theta g_{u u}.$$
where \( E_{pq} = r^2 (d_{p,q} + \frac{1}{2} h_{pq,u}) \), \( E_{up} = r^2 d_{p,u} - r^2 (d^a d^a)_{p} \) and \( \ast \Gamma_{pq}^{m} = \frac{1}{2} h^{m} (h_{pq,u} + h_{pu,q} - h_{pq,u}) \) and \( C_{pqnm} \) corresponding Weyl tensor of transverse space. Weyl tensors are become more simpler than the results of any arbitrary dimensions \( D > 4 \) \([11, 26]\).

Then, we can easily determine the obligatory conditions for types and sub-types of the vacuum RT solution by using Table 1.

- Vacuum RT solution will be sub-type Type II(a) if the Weyl scalar \( \Psi_{2S} = 0 \), which yields:
  \[ d_{p}r^p = 0. \]  

- Vacuum RT solution will be sub-type Type II(b) if the Weyl scalar \( \Psi_{2T(i)} = 0 \) which gives:
  \[ d_{p}q^p = -d^m \ast T^{m}_{pq} \] and \( d_{p}d_{p} = 0. \)

- Vacuum RT solution will be sub-type Type II(c) if the Weyl scalar \( \Psi_{2j,k} = 0 \), which yields:
  \[ \bar{C}_{pqnm} = 0. \]

- Vacuum RT solution will be sub-type Type II(d) if the Weyl scalar \( \Psi_{2j} = 0 \), which yields:
  \[ d_{p,q} = \frac{1}{2} u^m (E_{pm} - E_{qm}). \]

- Vacuum RT solution will be Type III (equivalently sub-type Type II(abcd)) if the above Weyl scalars vanish and equations (22)-(25) are satisfied simultaneously. Additionally, the spacetime becomes Type III(a) because the Weyl scalar \( \Psi_{1T} \) is obtained zero with these conditions.

- Vacuum RT solution will be sub-type Type III(b) and Type N if the Weyl scalar \( \Psi_{2j,k} = 0 \), which yields:
  \[ d_{p,q}h_{mn} = 0, \quad E_{pq[m]d_{q}] = 0,} \]
  \[ (h^{m} E_{pq[m]n}) = -h^{m} E_{pq[m]n}. \]

- Vacuum RT solution will be Type O if all above conditions are satisfied with the Weyl scalar \( \Psi_{2j} = 0 \), which yields:
  \[ d_{p}d_{p,u} = 0, \quad E_{pm}[u]d_{m} = 0, \]
  \[ (h^{m} E_{pm}[u]) = -h^{m} E_{pm}[u]. \]

Final condition requires the coefficient of \( d_{p} \) independent of the parameter \( u \) (\( d_{p}(u, x) \to d_{p}(x) \)).

The algebraic classification of RT geometry and obligatory conditions are summarized at Table 2 for primary WAND k. Various combinations of sub-types can be obtained by using the neccassary conditions such as Type II(ad) occur if equations (22) and (25) are satisfied simultaneously.

The secondary alignment Types Ii and IIi arise when the equation (21) vanishes which yields \( Ar^4 + Br^3 - Cr + D = 0 \) where

\[ A = -8d^m d_m d_p d_q + d_{p,q} d_{p,q} + 5d^m d_m \]
\[ B = -(2d_{p}d_{q} d^a d^a_{p} - d_{p,q} d_{p,q} + 4d^m d_m \ast T^{m}_{pq} \]
\[ + (h^{m} d_m )_{(p}h_{q)} + h_{pq}d^m d^m d^m \ast T^{m}_{pq} + (d^m d^m)_{p} \]
\[ - (d^m d_{q})_{u} - d_{p}(d^m d_{q})_{u} \]
\[ C = E_{pq[p}d_{q]} \]
\[ D = h_{pq}h^{m} E_{pq[m]u} \ast T^{m}_{pq} + 2h_{pq}h^{m} E_{pq[m]u}. \]

In addition, if the equations (22-25) are satisfied with \( d_{p}(u, x) \to d_{p}(x) \), the spacetime becomes Type III, for the secondary WAND \( \ell \).

Table 1. Algebraic classification of the RT geometry for the primary and secondary WANDs k, \( \ell \) \([11]\).

<table>
<thead>
<tr>
<th>Types</th>
<th>Vanishing Weyl Scalar</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( \Psi_{2j,k} = 0 ), ( \Psi_{2j} = 0 ), ( \Psi_{1T} = 0 ), ( \Psi_{3T} = 0 )</td>
</tr>
<tr>
<td>II(a)</td>
<td>( \Psi_{2j,k} = 0 ), ( \Psi_{2j} = 0 ), ( \Psi_{2T(i)} = 0 )</td>
</tr>
<tr>
<td>II(b)</td>
<td>( \Psi_{2j,k} = 0 ), ( \Psi_{2j} = 0 ), ( \Psi_{3T} = 0 )</td>
</tr>
<tr>
<td>II(c)</td>
<td>( \Psi_{2j,k} = 0 ), ( \Psi_{2j} = 0 ), ( \Psi_{2T(i)} = 0 )</td>
</tr>
<tr>
<td>II(d)</td>
<td>( \Psi_{2j,k} = 0 ), ( \Psi_{2j} = 0 ), ( \Psi_{3T} = 0 )</td>
</tr>
<tr>
<td>III(a)</td>
<td>( \Psi_{2j,k} = 0 ), ( \Psi_{2j} = 0 ), ( \Psi_{2T(i)} = 0 ), ( \Psi_{2j,k} = 0 ), ( \Psi_{3T} = 0 )</td>
</tr>
<tr>
<td>III(b)</td>
<td>( \Psi_{2j,k} = 0 ), ( \Psi_{2j} = 0 ), ( \Psi_{3T} = 0 )</td>
</tr>
<tr>
<td>N</td>
<td>( \Psi_{2j,k} = 0 ), ( \Psi_{2j} = 0 ), ( \Psi_{2T(i)} = 0 )</td>
</tr>
<tr>
<td>Ii</td>
<td>( \Psi_{2j,k} = 0 ), ( \Psi_{2j} = 0 ), ( \Psi_{3T} = 0 )</td>
</tr>
<tr>
<td>IIIi</td>
<td>( \Psi_{2j,k} = 0 ), ( \Psi_{2j} = 0 ), ( \Psi_{2T(i)} = 0 ), ( \Psi_{2j,k} = 0 ), ( \Psi_{3T} = 0 )</td>
</tr>
<tr>
<td>D</td>
<td>( \Psi_{2j,k} = 0 ), ( \Psi_{2j} = 0 ), ( \Psi_{3T} = 0 ), ( \Psi_{3T} = 0 )</td>
</tr>
</tbody>
</table>
Obligatory conditions for the primary and secondary WANDs and $\ell$.

4. Conclusion

We revisited the most general Robinson-Trautman metric in higher dimensions. By defining the most natural null frames, components of Weyl scalar were given at large $D$. Because $\Psi_{1ijk}$ and $\Psi_{1ijk}$ vanished, RT spacetime is Type I(b) or more special. Additionally, vanishing components of Weyl scalar are obligatory conditions for the primary and secondary WANDs $k$ and $\ell$. In future studies, by solving field equations, the property of the limitation method can be fully obtained.

Declaration


Conflict of Interest: The authors have declared no conflicts of interest.

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References


<table>
<thead>
<tr>
<th>Types</th>
<th>Obligatory Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>always</td>
</tr>
<tr>
<td>I(a)</td>
<td>always</td>
</tr>
<tr>
<td>I(b)</td>
<td>always</td>
</tr>
<tr>
<td>II</td>
<td>$d_p d_p = 0$</td>
</tr>
<tr>
<td>II(a)</td>
<td>$d_{ij} = -d_{ij} \Gamma^{mn}_{mq}$ and $d_p d_q = 0$</td>
</tr>
<tr>
<td>II(b)</td>
<td>$d_{[p,q]} = \frac{1}{\sqrt{2}} (E_{pn} - E_{qm})$</td>
</tr>
<tr>
<td>III</td>
<td>all above conditions</td>
</tr>
<tr>
<td>III(a)</td>
<td>all above conditions</td>
</tr>
<tr>
<td>III(b) and N</td>
<td>all above conditions , $d_{ij} = 0$, $E_{ij} = 0$, $(h_{ns} E_{sk})<em>{ij} = 0$, $d</em>{p,q} = 0$</td>
</tr>
<tr>
<td>O</td>
<td>$A = B = C = D = 0$</td>
</tr>
<tr>
<td>I, II</td>
<td>$d_{p,u} = 0 \leftrightarrow d_p (u, x) \rightarrow d_p (x)$</td>
</tr>
<tr>
<td>III$_h$</td>
<td>$d_{p,u} = 0 \leftrightarrow d_p (u, x) \rightarrow d_p (x)$</td>
</tr>
</tbody>
</table>

Table 2. Algebraic classification of the vacuum RT spacetime with obligatory conditions for the primary and secondary WANDs $k$ and $\ell$. 


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