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Dynamics of a Mathematical Model of Legionnaires' Disease

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Research Article

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ABSTRACT

Legionnaires' disease is a very serious type of pneumonia (lung infection) caused by bacteria called Legionella. In this research work, a mathematical model for the transmission dynamics of Legionnaires' disease is developed. Mathematical analysis is carried out to gain relevant insight on the basic features of the model. Some of the findings of this research indicate the existence of a legionnaire-free equilibrium, which was later shown to be globally stable, provided the control reproduction number is less than one. Further analysis showed legionnaires' disease to be endemic among the human population, which we proved to be globally asymptotically stable whenever the effective basic reproduction number is greater than unity and unstable whenever the effective basic reproduction number is less than unity. Furthermore, administering effective treatments to humans exposed to Legionnaires' disease should be prioritized, as shown in the simulation results.

1. Introduction

Legionnaires' disease (LD) is a chronic type of pneumonia, and recent studies have revealed that it can be transmitted from environment to humans and from humans to humans when contaminated aerosols (e.g., mist droplets containing the bacteria) are inhaled, which manifest similar symptoms as pneumonia [1]. However, the aerosols containing legionella bacteria have been produced via channels such as water from contaminated swimming pools, domestic hot-water systems, respiratory therapy devices, cooling towers, fountains, and other utilities where people make use of public water supplies [2]. Similar signs and symptoms found in pneumonia, such as fever, cough, and chest pain, are also observed in patients with Legionnaires' disease [3]. The fundamental cause of Legionnaires' diseases is the legionella bacterium, which has been described as a water-based organism that usually leads to infection when inhaled by individuals in an aerosol form [4]. It has been identified that the physical and chemical properties of water, the occurrence of heavy metals, and mineral contents are the major factors contributing to the survival of legionella [4].

This disease affects almost all age groups, most especially people who are advanced in age, in particular those diagnosed with severe pulmonary, renal, and cardiac diseases [4]. Multiple organs are affected, especially when patients are immune compromised [5]. So far, no reliable clinical laboratories or radiological parameters can distinguish between Legionnaires' disease and other pneumonia infections [4]. Therefore, it is imperative to always confirm clinical doubts with special laboratory tests (culture, detection of antigen in respiratory secretion or urine, serologic tests, and molecular diagnostics) [6].

Several species (spp) of legionella have been discovered to exist everywhere in nature, such as in soil and water. These legionella species are present in human-made water distribution systems, decorative fountains, cooling towers, and hot water tanks [7]. It is found that risk factors are high, especially in people with health conditions such as severe lung infection or people who are immune compromised [9]. Le¬gionnaires' disease was initially detected in 1976 in the USA, where two hundred and twenty-one (221) cases were recorded and thirty-four (34) infected persons died [3]. This upsurge has been noted as one of the largest community-associated outbreaks of Legionnaires' disease in the United States. It was later that this resurgence was found to be caused by the cooling system of the hosting hotel, and a bacterium categorized as Legionella pneumophila serogroup 1 was subsequently isolated from four (4) persons [8]. In 2015, a scientific investigation was conducted to systematically find out the cause of the outbreak that led to the deaths of sixteen (16) patients and one hundred and twenty-eight (128) patients that needed to be hospitalized.

1.1. Model Formulation

We formulate a mathematical model in which the number of people is classified into adults and children. The variables $N_a(t)$ and $N_c(t)$ denote all children and adults put together, respectively, at time t. Table 1 specifically shows the classification of the total population into the following mutually

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exclusive compartments:$(E_a(t), E_c(t))$ denote susceptible children and adults, respectively; $(I_{am}(t), I_{cm}(t))$ represent children and adults with Legionnaires' at a latent state; $(I_{as}(t), I_{cs}(t))$ are adults and children with mild disease; $(R_a(t), R_c(t))$ denote severe infections in both adults and children; represent the population of adults and children who have recovered from Legionnaire's disease. Hence, we have that

$$\begin{aligned} N_c &= S_c + E_c + I_{cm} + I_{cs} + R_c \\ N_a &= S_a + E_a + I_{am} + I_{as} + R_a \\ N &= N_c + N_a \end{aligned} \quad (1)$$

On the other hand, susceptible adults and children contract Legionnaires' disease when exposed to an infected human population with the force of infection given by

$$\lambda = \frac{\beta\left(I_{as} + I_{am} + \eta\left(I_{cs} + \eta_c I_{cm}\right)\right)(1-\theta)}{N} \quad (2)$$

where λ is the disease force of infection.

As shown in Figure 1, recruitment rates π_a, π_c for the child and adult populations are different**.** We assume that a slight percentage φ_a, φ_c of adults and children who have latent infections have moved to the mild class of infections as a result of public awareness (at rates α_a, α_c respectively). The remainder $(1-\varphi_a)$, $(1-\varphi_c)$ of infected adults and children develop serious infections at rates χ_a, χ_c that are higher, respectively. The parameter β denotes the probability that adults and children acquire infection per contact with infected persons. The class of adults at mild and severe stages of infection is reduced due to recovery at rates σ_{am}, σ_{as} respectively, while children at mild and severe stages recover at rates σ_{cm}, σ_{cs} respectively. Adults and children with mild and severe infection suffer additional disease-induced deaths at rates δ_{cm}, δ_{cs}, δ_{am}, δ_{as}. However, ω_a, ω_c denotes the rate at which recovered adults and children revert to susceptible classes while θ represents public enlightenment awareness. q_a , q_c represent the proportion of recovered humans that clear all the bacteria from the body while $(1-q_a)(1-q_c)$ is a proportion of those that still carry the bacteria.

Table 1. Description of the variables and parameters of the Legionnaires' model

Variables	Descriptions
S_c, S_a	Number of susceptible children and adults
E_c, E_a	Number of exposed children and adults
I_{cm}, I_{am}	Number of children and adults with mild infection
I_{cs}, I_{as}	Number of children and adults with severe infection
R_c, R_a	Recovered children and adult population
N_c, N_a	Total number of children and adult respectively
N	Total human population
Parameters	Descriptions
π_c, π_a	Human recruitment rate for children and adults
μ_c, μ_a	Natural death rate for children and adults
β	Transmission probability per contact for adults and children
α_c, α_a	Rate of progression from exposed children and adults' class to mild phase of infection
φ_c, φ_a	Fraction of exposed human (children and adults) who become infected at mild stages
$(1-\varphi_c)(1-\varphi_a)$	Remaining fraction of exposed human (children and adults) who acquire severe infection
χ_c, χ_a	Progression rate to severe stages of infection from mild stage for children and adult
δ_{am}, δ_{as}	Death rates due to infection for adults at mild and severe stages
δ_{cm}, δ_{cs}	Death rates due to infection for children at mild and severe stages
σ_{cm}, σ_{cs}	Recovery rates for children having mild and severe infection
σ_{am}, σ_{as}	Recovery rates for adult having mild and severe infection
γ	Growth and maturation rate
q_c, q_a	Proportion of recovered children and adults who clear all the bacteria from the body
$(1-q_c)(1-q_a)$	Proportion of those that still carry the bacteria
ω_c, ω_a	Reversion rate from recovered class to susceptible class for children and adults
θ	Public health awareness rate
η	Modification parameter
η_c	Modification parameters for children and adults due to infection

The equation below describes the Legionnaires' transmission dynamics.

$$\begin{aligned} \frac{dS_c}{dt} &= \pi_c + \omega_c R_c - \left(\lambda + \gamma + \mu_c\right) S_c \\ \frac{dE_c}{dt} &= \lambda S_c - \left(\alpha_c + \mu_c\right) E_c \end{aligned} \quad (3)$$

$$\frac{dI_{cm}}{dt} = \alpha_c \varphi_c E_c + (1-q_c)\sigma_{cs} I_{cs} - \left(\sigma_{cm} + \mu_c + \delta_{cm} + \chi_c\right) I_{cm}$$

$$\frac{dI_{cs}}{dt} = \alpha_c (1-\varphi_c) E_c + \chi_c I_{cm} - \left(\sigma_{cs} + \delta_{cs} + \mu_c\right) I_{cs}$$

$$\frac{dR_c}{dt} = \sigma_{cm} I_{cm} + q_c \sigma_{cs} I_{cs} - \left(\omega_c + \gamma + \mu_c\right) R_c$$

$$\frac{dS_a}{dt} = \pi_a + \omega_a R_a - \left(\lambda + \mu_a\right) S_a + \gamma S_c$$

$$\frac{dE_a}{dt} = \lambda S_a - \left(\alpha_a + \mu_a\right) E_a$$

$$\frac{dI_{am}}{dt} = \alpha_a \varphi_a E_a + (1-q_a)\sigma_{as} I_{as} - \left(\sigma_{am} + \mu_a + \delta_{am} + \chi_a\right) I_{am}$$

$$\frac{dI_{as}}{dt} = \alpha_a (1-\varphi_a) E_a + \chi_a I_{am} - \left(\sigma_{as} + \delta_{as} + \mu_a\right) I_{as}$$

$$\frac{dR_a}{dt} = \sigma_{am} I_{am} + q_a \sigma_{as} I_{as} - \left(\omega_a + \mu_a\right) R_a + \gamma R_c$$

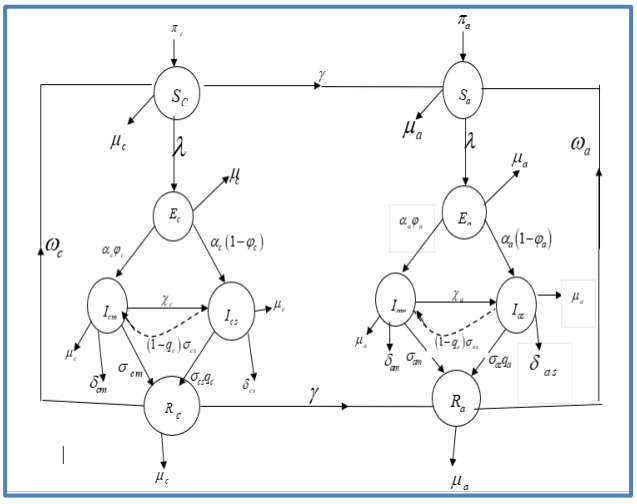


Fig 1. Flow Diagram of Legionnaires'

1.2. Fundamental Properties of Model Equations

For the Legionnaires' model to be epidemiologically meaningful, we can establish that all associated variables are non-negative for all (t) time. This can be proved by showing that all solutions of the model (3.0) with non-negative initial data will continue to be positive for $t>0$. Thus, the feasible region for the model is given by Δ.

Lemma 1.1: There exists a region Δ, where the model (3) is positively invariant and bounded in R_+^{10}.

***Proof*:** On adding the equations of system (3), we get the total population as

$$\frac{dN}{dt}=\pi_c+\pi_a-\mu_cN_c-\mu_aN_a-\left(\delta_{cm}I_{cm}+\sigma_{cm}I_{cm}+\delta_{am}I_{am}+\delta_{as}I_{as}\right) \quad (4)$$

By the standard comparison theorem, we have

$$\frac{dN}{dt}\le\pi_c+\pi_a-\mu_cN_c-\mu_aN_a \quad (5)$$

$$\frac{dN}{dt}\le\pi_c+\pi_a-\mu_hN$$

where $\mu_h=\min\left\{\mu_c,\mu_a\right\}$. (6)

By using the method of integrating factor, we have

$$N(t)=N(0)e^{-\mu_ht}+\frac{\pi_c+\pi_a}{\mu_h}\left(1-e^{-\mu_ht}\right), \quad (7)$$

Therefore, if $N(0)\le\frac{\pi_c+\pi_a}{\mu_h}$, then

$$N(t)\le\frac{\pi_c+\pi_a}{\mu_h} \quad \text{for all } t>0.$$

Therefore, Δ is positively invariant and contains the solution of the model. This implies that the solutions to the Legionnaires' model are non-negative and are contained in the feasible region

$$\Delta=\left\{\left(S_c,E_c,I_{cm},I_{cs},R_c,S_a,E_a,I_{am},I_{as},R_a\right)\right\}$$

Hence, the dynamics of system (3) is biologically useful and necessary to study in Δ.

Lemma 1.2: Let the initial data for system (3) be $S_c(0)>0,E_c(0)>0,I_{cm}(0)>0,I_{cs}(0)>0,R_c(0)>0,$
$S_a(0)>0,E_a(0)>0,I_{am}(0)>0,I_{as}(0)>0,R_a(0)>0,$

then, the solutions of (S_c, E_c, I_{cm}, I_{cs}, R_c, S_a, E_a, I_{am}, I_{as}, R_a) to the model equation (3) with starting conditions will remain positive for all time $t > 0$.

***Proof*:**

The initial conditions of the model are assumed to be non-negative, and the procedure in [18] will be adopted to prove that the model solutions are positive.

$$\text{Let } t_1=\sup\begin{Bmatrix}t>0;S_c(0)>0,E_c(0)>0,I_{cm}(0)>0,I_{cs}(0)>0,R_c(0)>0,\\S_a(0)>0,E_a(0)>0,I_{am}(0)>0,I_{as}(0)>0,R_a(0)>0,\end{Bmatrix}>0 \quad (8)$$

From the first equation in system (3)

$$S_c=\pi_c+\omega_cR_c-\lambda S_c-(\gamma+\mu_c)S_c \quad (9)$$

this implies

$$S_c\ge\pi_c-(\lambda+\gamma+\mu_c)S_c \quad (10)$$

Adopting the integrating factor procedure gives

$$\frac{d}{dt}\left\{S_c(t)\exp\left(\gamma+\mu_c+\int_0^{t_1}\lambda(\tau)d\tau\right)\right\}=\pi_c\exp\left(\gamma+\mu_c+\int_0^{t_1}\lambda(\tau)d\tau\right) \quad (11)$$

Integrating from 0 to, we have

$$S_c(t)\exp\left(\gamma+\mu_c+\int_0^{t_1}\lambda(\tau)d\tau\right)-S_c(0)\ge\int_0^{t_1}\pi_c\exp\left(\gamma+\mu_c+\int_0^{t_1}\lambda(\tau)d\tau\right) \quad (12)$$

$$S_c(t_1)\ge S_c(0)\exp\left(-\left(\gamma+\mu_c\right)t-\int_0^{t_1}\lambda(\tau)d\tau\right)+\exp\left(-\left(\gamma+\mu_c\right)t-\int_0^{t_1}\lambda(\tau)d\tau\right)\times\int_0^{t_1}\pi_c\left[\exp\left(\left(\gamma+\mu_c\right)y+\int_0^{y}\lambda(\tau)d\tau\right)\right]dy>0 \quad (13)$$

A similar procedure as shown above can be adopted to prove that the remaining solutions of system (3) are positive for all time $t > 0$.

$E_c(t)>0,I_{cm}(t)>0,I_{cs}(t)>0,R_c(t)>0,S_a(t)>0,E_a(t)>0,I_{am}(t)>0,$
$I_{as}(t)>0,R_a(t)>0,$ for all time $t>0$

2. Model Analysis

2.1. Control Reproduction Number of the Legionnaires-Free Equilibrium (LFE)

Using the approach in [10], system (3) has a distinct Legionnaires-free equilibrium defined as

$$\xi_0=\left(S_c,E_c,I_{cm},I_{cs},R_c,S_a,E_a,I_{am},I_{as},R_a\right)=\left(\frac{\pi_c}{(\gamma+\mu_c)},0,0,0,0,\frac{\pi_a(\gamma+\mu_c)+\gamma\pi_c}{\mu_a(\gamma+\mu_c)},0,0,0,0\right) \quad (14)$$

The control reproduction number (R_E) was previously computed in [22] to give

$$R_E = \frac{R_{0,4}^a + R_{0,1}^c + \sqrt{\left(R_{0,1}^c - R_{0,4}^a\right)^2 + 4R_{0,4}^c R_{0,1}^a}}{2}, \tag{15}$$

Where; $$R_{0,1}^c = \frac{\beta\eta\left(1-\theta\right)\pi_c\mu_a\left[\eta_c\left((1-q_c)\sigma_{cs}\left(\alpha_c(1-\theta_c)\right)-k_4\alpha_c\varphi_c\right)+\left((\alpha_c\varphi_c\chi_c)-k_3\alpha_c(1-\varphi_c)\right)\right]}{k_2\left[\pi_c\mu_a+\pi_a k_1+\pi_c\gamma\right]\left[k_3k_4-\left(1-q_c\right)\left(\sigma_{cs}\chi_c\right)\right]}, \tag{16}$$

$$R_{0,4}^c = \frac{\beta\left(1-\theta\right)\pi_c\mu_a\left[\left(k_9(-\alpha_a\varphi_a)-\left((1-q_a)\sigma_{as}\right)\left(\alpha_a(1-\varphi_a)\right)\right)+\left(k_8\left(-\alpha_a(1-\varphi_a)\right)+\alpha_a\varphi_a\chi_a\right)\right]}{k_7\left[\pi_c\mu_a+\pi_a k_1+\pi_c\gamma\right]\left[k_8k_9+\left(1-q_a\right)\left(\sigma_{as}\chi_a\right)\right]}, \tag{17}$$

$$R_{0,1}^a = \frac{\beta\eta\left(1-\theta\right)\left(\pi_a k_1+\pi_c\gamma\right)\left[\eta_c\left((1-q_c)\sigma_{cs}\left(\alpha_c(1-\theta_c)\right)-k_4\alpha_c\varphi_c\right)+\left((\alpha_c\varphi_c\chi_c)-k_3\alpha_c(1-\varphi_c)\right)\right]}{k_2\left[\pi_c\mu_a+\pi_a k_1+\pi_c\gamma\right]\left[k_3k_4-\left(1-q_c\right)\left(\sigma_{cs}\chi_c\right)\right]}, \tag{18}$$

$$R_{0,4}^a = \frac{\beta\left(1-\theta\right)\left(\pi_a k_1+\pi_c\gamma\right)\left[\left(k_9(-\alpha_a\varphi_a)-\left((1-q_a)\sigma_{as}\right)\left(\alpha_a(1-\varphi_a)\right)\right)+\left(k_8\left(-\alpha_a(1-\varphi_a)\right)+\alpha_a\varphi_a\chi_a\right)\right]}{k_7\left[\pi_c\mu_a+\pi_a k_1+\pi_c\gamma\right]\left[k_8k_9+\left(1-q_a\right)\left(\sigma_{as}\chi_a\right)\right]}, \tag{19}$$

2.2. Analysis of the Reproduction Numbers, $R_{0,1}^c, R_{0,4}^c, R_{0,1}^a$ and $R_{0,4}^a$

Recall the quantitative parameter, $R_{0,1}^c$, we explore the after-effect of control measures for individuals on the mild and severe phases of Legionnaires' disease and the change in the movement from the mild to the severe phase of infection on the control of legionnaires disease in the human population.

From (16), we have

$$R_{0,1}^c = \frac{\beta\eta\left(1-\theta\right)\pi_c\mu_a\left[\eta_c\left((1-q_c)\sigma_{cs}\left(\alpha_c(1-\theta_c)\right)-k_4\alpha_c\varphi_c\right)+\left((\alpha_c\varphi_c\chi_c)-k_3\alpha_c(1-\varphi_c)\right)\right]}{k_2\left[\pi_c\mu_a+\pi_a k_1+\pi_c\gamma\right]\left[k_3k_4-\left(1-q_c\right)\left(\sigma_{cs}\chi_c\right)\right]}$$

$$\lim_{\sigma_{cm}\to\infty} R_{0,1}^c = 0 \tag{20}$$

$$\lim_{\chi_c\to\infty} R_{0,1}^c > 0 \tag{21}$$

It is evident from ($R_{0,1}^a$) that

$$\lim_{\sigma_{cm}\to\infty} R_{0,1}^a = 0 \tag{22}$$

Similarly,

$$\lim_{\sigma_{as}\to\infty} R_{0,4}^a > 0 \tag{23}$$

$$\lim_{\sigma_{am}\to\infty} R_{0,4}^a > 0 \tag{24}$$

$$\lim_{\chi_a\to\infty} R_{0,4}^a > 0 \tag{25}$$

Therefore, a Legionnaires' control program that leads to high control rates for various treatments ($\sigma_{cm}, \sigma_{cs}, \sigma_{am}, \sigma_{as}\to\infty$) and a condition whereby the movement of children and adults changes from the mild to the severe phase of infection($\chi_c, \chi_a\to\infty$) are high (accounting for rapid progression) and can lead to effective Legionnaires' control if the right-hand sides of (23–25) can be reduced below one.

From $\lim_{\sigma_{cm}\to\infty} R_{0,1}^c = 0$ and $\lim_{\sigma_{cm}\to\infty} R_{0,1}^a = 0$, a near-total eradication of Legionnaires' is achievable.

Here, the effective treatment is to focus on treating individuals in the mild stage of infection since they contribute less to the spread of the disease and have a high treatment rate ($\sigma_{cm}\to\infty$). This is very effective since subsequent disease progressions proceed from this mild stage of the Legionnaires' infection.

Solving for the partial derivatives of $R_{0,1}^c, R_{0,4}^c, R_{0,1}^a, R_{0,4}^a$ with respect to the parameters under investigation, ($\sigma_{cm}, \sigma_{cs}, \sigma_{am}, \sigma_{as}, \chi_c$ and χ_a) reveals the influence of these parameters in reducing the spread of Legionnaires' disease in the community. Thus,

$$\frac{\partial R_{0,1}^c}{\partial\sigma_{am}} = \frac{\partial R_{0,1}^c}{\partial\sigma_{as}} = \frac{\partial R_{0,1}^c}{\partial\chi_a} = 0 \tag{26}$$

$$\frac{\partial R_{0,4}^c}{\partial\sigma_{am}} < 0 \tag{27}$$

$$\frac{\partial R_{0,4}^c}{\partial\sigma_{cm}} = \frac{\partial R_{0,4}^c}{\partial\sigma_{cs}} = \frac{\partial R_{0,4}^c}{\partial\chi_a} = 0 \tag{28}$$

$$\frac{\partial R_{0,1}^a}{\partial\sigma_{am}} = \frac{\partial R_{0,1}^a}{\partial\sigma_{am}} = \frac{\partial R_{0,1}^a}{\partial\chi_a} = 0 \tag{29}$$

$$\frac{\partial R_{0,4}^a}{\partial\sigma_{am}} < 0 \tag{30}$$

$$\frac{\partial R_{0,4}^{a}}{\partial \sigma_{cm}} = \frac{\partial R_{0,4}^{a}}{\partial \sigma_{cs}} = \frac{\partial R_{0,4}^{a}}{\partial \chi_{c}} = 0 \tag{31}$$

Clearly, it follows from (27) and (30) that the partial derivatives are less than zero, categorically. Therefore, the actual treatment rate of Legionnaires' (for both mild and severe phases of infection) and the swift movement beginning from the severe phase to the recovered phase of Legionnaire manifestation will have the optimal impact on dropping the burden of Legionnaires' in the human community, notwithstanding the values of the other parameters in the expressions on the right-hand sides of (27) and (30).

Lemma 1.3 Effective treatment rates (σ_{cm} *and* σ_{am})for both children and adults) for mild and severe phases of legionnaire and at a faster rate of progression from severe to the recovered phases of the disease (σ_{as}) will have an optimum impact in dropping the Legionnaires' problem in human community, notwithstanding the values of the other parameters in the actual reproduction number.

It therefore, can be shown that $\frac{\partial R_{0,1}^{c}}{\partial \sigma_{cm}} < 0$

$$\text{if} \quad \left(1-q_c\right)\sigma_{cs}\alpha_c\left(1-\theta\right) > \left(\left(\sigma_{cs}+\delta_{cs}+\mu_c\right)\alpha_c\varphi_c\right)\sigma_{cs}+\delta_{cs}+\mu_c \quad \text{and} \quad \left(\sigma_{cm}+\delta_{cm}+\mu_c+\chi_c\right)\left(\sigma_{cs}+\delta_{cs}+\mu_c\right) > \left(1-q_c\right)\sigma_{cs}\chi_c \tag{32}$$

$$\frac{\partial R_{0,1}^{c}}{\partial \sigma_{cs}} < 0 \text{ if } \alpha_c\varphi_c > \alpha_c\varphi_c\left(1-q_c\right)\alpha_c\left(1-\theta\right),\ \left(1-q_c\right)\sigma_{cs}\alpha_c\left(1-\theta\right) > \left(\sigma_{cs}+\delta_{cs}+\mu_c\right)\alpha_c\varphi_c \text{ and}$$

$$\left(\sigma_{cm}+\delta_{cm}+\mu_c+\chi_c\right)\left(\sigma_{cs}+\delta_{cs}+\mu_c\right) > \left(1-q_c\right)\sigma_{cs}\chi_c,\ \sigma_{cm}+\delta_{cm}+\mu_c+\chi_c > \left(1-q_c\right)\chi_c \tag{33}$$

That is, a Legionnaires' disease control program becomes necessary if the control (treatment) rate of individuals manifesting symptoms in the severe phases of infection is greater than the control rate of individuals manifesting symptoms in the mild stage of Legionnaires' disease. This is because people at a mild stage of infection contribute less to the spread of the disease. Thus, it is certain that rapid changes in the movement from the primary to the advanced phase of the Legionnaires' disease manifestation will have a notable effect on dropping the Legionnaires' disease threat in the community when the changes in control of individuals in the severe stage of the disease quadruple the changes in individuals in the mild stage of the disease. A likely approach is applied to the remaining basic reproduction numbers. It follows from (28) and (29), that a faster movement from the mild stage of Legionnaires' disease to the severe stage could show an insignificant effect on Legionnaires' disease control in the human population. The breakdown in this section examines the effect of changes from the mild and severe phases of Legionnaires' disease through the changes in control measures administered to individuals in the mild and severe phases of the infection in the human population.

2.3. Existence of Legionnaires -Endemic Equilibrium Point (LEE)

Lemma 1.4: A distinct (positive) Legionnaires-endemic equilibrium exist for the special case if $\omega_c = \omega_a = \gamma = 0$ whenever $R_E > 1$.

Proof: To establish the existence of the Legionnaires- endemic equilibrium point of the system (3) with the condition $\omega_c = \omega_a = \gamma = 0$ we use the approach in [18]. Setting the LEE of the model to be

$$\xi^{**} = \left(S_c^{**}, E_c^{**}, I_{cm}^{**}, I_{cs}^{**}, R_c^{**}, S_a^{**}, E_a^{**}, I_{am}^{**}, I_{as}^{**}, R_a^{**},\right) \tag{34}$$

The equation in system (3) with $\omega_c = \omega_a = \gamma = 0$ is solved in terms of the forces of infection at steady state.

$$S_c = \frac{\pi_c}{\left(\lambda+\mu_c\right)},\ E_c = \frac{\lambda\pi_c}{\left(\lambda+\mu_c\right)k_2},\ I_{cm} = \frac{\left(\varphi_c q_c\sigma_{cs} + k_2\varphi_c - \varphi_c\sigma_{cs} - q_c\sigma_{cs} + \sigma_{cs}\right)\lambda\alpha_c\pi_c}{k_2\left(\lambda+\mu_c\right)\left(\chi_c q_c\sigma_{cs} + k_3k_4 - \chi_c\sigma_{cs}\right)} \tag{35}$$

$$I_{cs} = \frac{\lambda\alpha_c\pi_c\left(-k_3\varphi_c + \chi_c\varphi_c + k_3\right)}{k_2\left(\lambda\chi_c q_c\sigma_{cs} + \chi_c\mu_c q_c\sigma_{cs} + \lambda k_3k_4 - \lambda\chi_c\sigma_{cs} + k_3k_4\mu_c - \chi_c\mu_c\sigma_{cs}\right)},$$

$$R_c = \frac{\left(\begin{matrix}-k_3\varphi_c q_c\sigma_{cs} + \chi_c\varphi_c q_c\sigma_{cs} + \varphi_c q_c\sigma_{cm}\sigma_{cs} + k_3 q_c\sigma_{cs} + k_4\varphi_c q_{cm} - \varphi_c\sigma_{cm}\sigma_{cs} - q_c\sigma_{cm}\sigma_{cs} \\ +\sigma_{cm}\sigma_{cs}\end{matrix}\right)\lambda\alpha_c\pi_c}{k_2\left(\lambda+\mu_2\right)k_5\left(\chi_c q_c\sigma_{cs} + k_3k_4 - \chi_c\sigma_{cs}\right)} \tag{36}$$

$$S_a = \frac{\pi_c}{k_6},\ E_a = \frac{\lambda\pi_c}{k_7\left(\lambda+\mu_c\right)k_2},\ I_{am} = \frac{\left(\varphi_a q_a\sigma_{as} + k_9\varphi_a - \varphi_a\sigma_{as} - q_a\sigma_{as} + \sigma_{cs}\right)\lambda\alpha_a\pi_a}{k_7\left(\lambda+\mu_c\right)\left(\chi_a q_a\sigma_{as} + k_8k_9 - \chi_a\sigma_{as}\right)},$$

$$I_{as} = \frac{\lambda\alpha_a\pi_a\left(-k_8\varphi_a + \chi_a\varphi_a + k_a\right)}{k_7\left(\lambda\chi_a q_a\sigma_{as} + \chi_a\mu_c q_a\sigma_{as} + \lambda k_8k_9 - \lambda\chi_a\sigma_{as} + k_8k_9\mu_c - \chi_a\mu_c\sigma_{as}\right)}, \tag{37}$$

$$R_a = \frac{\left(\begin{matrix}-k_8\varphi_a q_a\sigma_{as} + \chi_a\varphi_a q_a\sigma_{as} + \varphi_a q_a\sigma_{am}\sigma_{as} + k_8 q_a\sigma_{as} + k_9\varphi_a q_{am} - \varphi_a\sigma_{am}\sigma_{as} - q_a\sigma_{am}\sigma_{as}\\ +\sigma_{cm}\sigma_{cs}\end{matrix}\right)\lambda\alpha_a\pi_a}{k_7\left(\lambda+\mu_c\right)k_{10}\left(\chi_a q_a\sigma_{as} + k_8k_9 - \chi_a\sigma_{as}\right)}$$

Substituting the values of $I_{cm}, I_{cs}, I_{am}, I_{as},$ and N into the force of infection (λ), where

$$N = \frac{k_2k_7\left(\lambda+\mu_c\right)\left(\pi_c+\pi_a\right) - \lambda\left[k_7\left(\delta_{cm}D_1 + \delta_{cs}D_2\right) + \left(\delta_{am}D_3 + \delta_{as}D_4\right)\right]}{\mu k_2k_7\left(\lambda+\mu_c\right)} \tag{38}$$

$$\lambda = \frac{\beta\left[I_{as} + I_{am} + \eta\left(I_{cs} + \eta_c I_{cm}\right)\right]\left(1-\theta\right)}{N} \tag{39}$$

gives

$$\lambda\left\{\lambda\left[k_2k_7\left(\pi_c+\pi_a\right) - k_7\left(\delta_{cm}D_1 + \delta_{cs}D_2\right) + k_2\left(\delta_{am}D_3 + \delta_{as}D_4\right)\right] - \beta(1-\theta)\mu\left[k_7\left(D_4+D_3\right) + k_2\left(D_1+D_2\right)\right]\right\} = 0 \tag{40}$$

which implies either $\lambda = 0$ or

$$\begin{aligned}
&\lambda\left[k_2k_7\left(\pi_c+\pi_a\right) - k_7\left(\delta_{cm}D_1 + \delta_{cs}D_2\right) + k_2\left(\delta_{am}D_3 + \delta_{as}D_4\right)\right] - \beta\left(1-\theta\right)\mu\left[k_7\left(D_4+D_3\right) + k_2\left(D_1+D_2\right)\right] = 0\\
&\lambda\left[k_2k_7\left(\pi_c+\pi_a\right) - k_7\left(\delta_{cm}D_1 + \delta_{cs}D_2\right) + k_2\left(\delta_{am}D_3 + \delta_{as}D_4\right)\right] = \beta\left(1-\theta\right)\mu\left[k_7\left(D_4+D_3\right) + k_2\left(D_1+D_2\right)\right]\\
&\lambda = \frac{\beta\left(1-\theta\right)\mu\left[k_7\left(D_4+D_3\right) + k_2\left(D_1+D_2\right)\right]}{k_2k_7\left(\pi_c+\pi_a\right) - k_7\left(\delta_{cm}D_1 + \delta_{cs}D_2\right) + k_2\left(\delta_{am}D_3 + \delta_{as}D_4\right)}
\end{aligned} \tag{41}$$

As a result, there exists a Legionnaires-free equilibrium point at $\lambda = 0$ and a Legionnaires-endemic equilibrium point (LEE) at

$$\lambda = \frac{\beta\left(1-\theta\right)\mu\left[k_7\left(D_4+D_3\right) + k_2\left(D_1+D_2\right)\right]}{k_2k_7\left(\pi_c+\pi_a\right) - k_7\left(\delta_{cm}D_1 + \delta_{cs}D_2\right) + k_2\left(\delta_{am}D_3 + \delta_{as}D_4\right)} \tag{42}$$

where; $D_1 = k_4k_3 - \left(1-q_c\right)\sigma_{sc}\chi_c,\ D_2 = k_4k_3 - \left(1-q_a\right)\sigma_{as}\chi_a$

$$\begin{aligned}
D_3 &= \lambda\chi_c q_c\sigma_{cs} + \chi_c\mu_c q_c\sigma_{cs} + \lambda k_3k_4 - \lambda\chi_c\sigma_{cs} + k_3k_4\mu_c - \chi_c\mu_c\sigma_{cs}\\
D_4 &= \lambda\chi_a q_a\sigma_{as} + \chi_a\mu_c q_a\sigma_{as} + \lambda k_8k_9 - \lambda\chi_a\sigma_{as} + k_8k_9\mu_c - \chi_a\mu_c\sigma_{as}
\end{aligned} \tag{43}$$

2.4. Global Asymptotic Stability of Legionnaires-Free Equilibrium (LFE)

Lemma 1.5: The LFE defined by $\in_0$ is globally asymptotically stable in Δ if $R_E \le 1$.

Proof: To prove the global asymptotic stability of LFE, we follow the procedure adopted by [19].

Let $X = \left(S_c, R_c, S_a, R_a,\right)$ and $Z = \left(E_c, I_{cm}, I_{cs}, E_a, I_{am}, I_{as},\right)$, and writing the model equations of system (3) in the form. $\frac{dX}{dt} = H\left(X,0\right), \frac{dZ}{dt} = Q\left(X,0\right),$

Where $E_c = I_{cm} = I_{cs} = E_a = I_{am} = I_{as} = 0$ with $H\left(X,0\right)$ being the R.H.S of $S_c^*, R_c^*, S_a^*, R_a^*$ and $Q\left(X,Z\right)$ the R.H.S of $E_c^*, I_{cm}^*, I_{cs}^*, E_a^*, I_{am}^*, I_{as}^*$ The reduced form of system (3) represented by $\frac{dX}{dt} = H\left(X,0\right)$ is

$$\begin{aligned}
S_c^* &= \pi_c + \omega_c R_c - \left(\gamma+\mu_c\right)S_c\\
R_c^* &= -\left(\omega_c + \gamma + \mu_c\right)R_c\\
S_a^* &= \pi_a + \omega_a R_a + \gamma S_a - \mu_a S_a\\
R_a^* &= -\left(\omega_a + \mu_a\right)R_a + \gamma R_a
\end{aligned} \tag{44}$$

The equilibrium of the system (44) is thus derived as

$$X^* = \left(S_c^*, R_c^*, S_a^*, R_a^*\right) = \left[\frac{\pi_c}{\left(\gamma+\mu_c\right)}, 0,0,0,0, \frac{\pi_a\left(\gamma+\mu_c\right)+\gamma\pi_c}{\mu_a\left(\gamma+\mu_c\right)}, 0,0,0,0\right] \quad (45)$$

Solving the reduced system (44) and taking limit as $t \to \infty$ proves X^* is globally stable in Δ.

The solution for $R_c^*(t)$ in (44) is obtain as follows:

$$\begin{aligned} &\left[InR_c\right]_{R_c(0)}^{R_c(t)} = -\left(\omega+\gamma+\mu\right)t \\ &\frac{R_c(t)}{R_c(0)} = e^{-(\omega+\gamma+\mu)t} \\ &R_c(t) = R_c(0)e^{-(\omega+\gamma+\mu)t} \\ &\lim R_c(t) = 0 \\ &t \to \infty \end{aligned} \quad (46)$$

Solving for $S_c(t)$ yields

$$S_c(t) = S_c(0)e^{-(\gamma+\mu_c)} + \frac{\pi_c}{\left(\gamma+\mu_c\right)}\left[1-e^{-(\gamma+\mu_c)}\right] + \left[\int_0^\tau \omega_c R_c(\tau)e^{(\gamma+\mu_c)}d\tau\right]e^{-(\gamma+\mu_c)t} \quad (47)$$

By solving $S_a(t)$ and substituting, $S_c(t)$ we obtained that

$$\begin{aligned} &S_a(t) = S_a(0)e^{-\mu_a(t)} + \frac{\pi_a}{\mu_a}\left[1-e^{-\mu_a(t)}\right] + \frac{\gamma\pi_c}{\mu_a\left(\gamma+\mu_c\right)} + \left[\int_0^\tau \omega_a R_a e^{\mu_a(t)}d\tau\right]e^{-\mu_a(t)} \\ &\lim S_a(t) = \frac{\pi_a}{\mu_a} + \frac{\gamma\pi_c}{\mu_a\left(\gamma+\mu_c\right)} \\ &t \to \infty \end{aligned} \quad (48)$$

The solution for $R_a(t)$ from

$$R_a^* = -\left(\omega_a+\mu_a\right)R_a + \gamma R_c \quad (49)$$

gives

$$\text{Lim}_{t\to\infty} R_a(t) = 0 \quad (50)$$

The above solution implies that the asymptotic behaviors of system (3) are independent of starting conditions in the feasible region. This indicates a global convergence of the solution in Δ From [11], it becomes necessary to prove that $Q(X,Z)$ establishes a correspondence with the following conditions:

$$(1)\, H(X,0) = 0 \quad (51)$$

$$(11) Q(X,0) = D_z Q\left(X^*,0\right) - Q^*(X,Z), \text{where } Q(X,Z) \geq 0, \quad (52)$$

Differentiating $Q(X,0)$ in regards to the infected compartments and evaluating at, $\left(X^*,0\right)$ we obtain the Jacobian of $Q(X,0)$ written as

$$D_z Q\left(X^*,0\right)Z = \begin{bmatrix} -k_2 & \frac{\beta(\eta\eta_c)(1-\theta)S_c^*}{N} & \frac{\beta\eta(1-\theta)S_c^*}{N} & 0 & \frac{\beta(1-\theta)S_c^*}{N} & \frac{\beta(1-\theta)S_c^*}{N} \\ \alpha_c\varphi_c & \left(1-q_c\right)\sigma_{cs} & -k_3 & 0 & 0 & 0 \\ \alpha_c\left(1-\varphi_c\right) & -k_4 & \chi_c & 0 & 0 & 0 \\ 0 & \frac{\beta(\eta\eta_c)(1-\theta)S_a^*}{N} & \frac{\beta\eta(1-\theta)S_a^*}{N} & -k_7 & \frac{\beta(1-\theta)S_a^*}{N} & \frac{\beta(1-\theta)S_a^*}{N} \\ 0 & 0 & 0 & \alpha_a\varphi_a & \left(1-q_a\right)\sigma_{as} & -k_8 \\ 0 & 0 & 0 & \alpha_a\left(1-\varphi_a\right) & -k_9 & \chi_a \end{bmatrix} \quad (53)$$

Then we have

$$\hat{Q}(X,Z) = \begin{bmatrix} \frac{\beta(1-\theta)S_c^*}{N^*}\left(1-\frac{N^*}{S_c^*}\frac{S_c}{N}\right)\left[\eta\eta_c I_{cm}^* + \eta I_{cs}^* + I_{am}^* + I_{as}^*\right] \\ 0 \\ 0 \\ \frac{\beta(1-\theta)S_a^*}{N^*}\left(1-\frac{N^*}{S_a^*}\frac{S_a}{N}\right)\left[\eta\eta_c I_{cm}^* + \eta I_{cs}^* + I_{am}^* + I_{as}^*\right] \\ 0 \\ 0 \end{bmatrix} \quad (54)$$

Since $S_c^* = \frac{\pi_c}{\left(\gamma+\mu_c\right)}, S_a^* = \frac{\pi_a\left(\gamma+\mu_c\right)+\gamma\pi_c}{\mu_a\left(\gamma+\mu_c\right)}$ (55)

In D, $S_a \leq S_a^*, S_c \leq S_c^*$ and $N \leq N^*$ if the population is at equilibrium than we have

$$\left(1-\frac{N^*}{S_c^*}\frac{S_c}{N}\right) > 0, \left(1-\frac{N^*}{S_a^*}\frac{S_a}{N^*}\right) > 0 \quad (56)$$

This implies $\hat{Q}(X,Z) \geq 0$. Thus, from [11], it can be concluded that the LFE is globally asymptotically stable.

2.4. Global Asymptotic Stability of LEE

Lemma 1.6: The Legionnaires-endemic equilibrium of system (3) is globally asymptotically stable in Δ whenever $R_E > 1$.

Proof: Assume, $R_E > 1$ we shall use the common quadratic Lyapunov function to establish the global stability of the Legionnaires-endemic equilibrium, as illustrated in [20]. Let us examine

$$v\left(x_1, x_2, x_i, ..., x_n\right) = \sum_{i=1}^{n} \frac{c_i}{2}\left(x_i - x_i^*\right)^2 \quad (57)$$

$$\begin{aligned} v\left(S_c, E_c, I_{cm}, I_{cs}, R_c, S_a, E_a, I_{am}, I_{as}, R_a\right) &= \frac{1}{2}\left[\left(S_c - S_c^*\right) + \left(E_c - E_c^*\right) + \left(I_{cm} - I_{cm}^*\right) + \left(I_{cs} - I_{cs}^*\right) + \left(R_c - R_c^*\right)\right] \\ &+ \frac{1}{2}\left[\left(S_a - S_a^*\right) + \left(E_a - E_a^*\right) + \left(I_{am} - I_{am}^*\right) + \left(I_{as} - I_{as}^*\right) + \left(R_a - R_a^*\right)\right] \end{aligned} \quad (58)$$

The time- derivative of v gives

$$\begin{aligned} \frac{dv}{dt} &= \left[\left(S_c - S_c^*\right) + \left(E_c - E_c^*\right) + \left(I_{cm} - I_{cm}^*\right) + \left(I_{cs} - I_{cs}^*\right) + \left(R_c - R_c^*\right)\right] \times \frac{d}{dt}\left(S_c + E_c + I_{cm} + I_{cs} + R_c\right) \\ &+ \left[\left(S_a - S_a^*\right) + \left(E_a - E_a^*\right) + \left(I_{am} - I_{am}^*\right) + \left(I_{as} - I_{as}^*\right) + \left(R_a - R_a^*\right)\right] \times \frac{d}{dt}\left(S_a + E_a + I_{am} + I_{as} + R_a\right) \end{aligned} \quad (59)$$

From model of system (3), we have that

$$\frac{d}{dt}\left(S_c+E_c+I_{cm}+I_{cs}+R_c\right)=\pi_c-\mu_c\left(S_c+E_c+I_{cm}+I_{cs}+R_c\right)-\delta_{cm}I_{cm}-\delta_{cs}I_{cs}$$
$$\frac{d}{dt}\left(S_a+E_a+I_{am}+I_{as}+R_a\right)=\pi_a-\mu_a\left(S_a+E_a+I_{am}+I_{as}+R_a\right)-\delta_{am}I_{am}-\delta_{as}I_{as} \quad (60)$$

Substituting (63) into (62) gives

$$\frac{dv}{dt}=\left[\left(S_c-S_c^*\right)+\left(E_c-E_c^*\right)+\left(I_{cm}-I_{cm}^*\right)+\left(I_{cs}-I_{cs}^*\right)+\left(R_c-R_c^*\right)\right]\times\left(\pi_c-\mu_c\left(S_c+E_c+I_{cm}+I_{cs}+R_c\right)-\delta_{cm}I_{cm}-\delta_{cs}I_{cs}\right)$$
$$+\left[\left(S_a-S_a^*\right)+\left(E_a-E_a^*\right)+\left(I_{am}-I_{am}^*\right)+\left(I_{as}-I_{as}^*\right)+\left(R_a-R_a^*\right)\right]\times\left(\pi_a-\mu_a\left(S_a+E_a+I_{am}+I_{as}+R_a\right)-\delta_{am}I_{am}-\delta_{as}I_{as}\right) \quad (61)$$

Now assuming that

$$\pi_c=\mu_c\left(S_c+E_c+I_{cm}+I_{cs}+R_c\right)-\delta_{cm}I_{cm}-\delta_{cs}I_{cs}$$
$$\pi_a=\mu_a\left(S_a+E_a+I_{am}+I_{as}+R_a\right)-\delta_{am}I_{am}-\delta_{as}I_{as} \quad (62)$$

Substituting (62) into (61) yields

$$\frac{dv}{dt}=\left[\left(S_c-S_c^*\right)+\left(E_c-E_c^*\right)+\left(I_{cm}-I_{cm}^*\right)+\left(I_{cs}-I_{cs}^*\right)+\left(R_c-R_c^*\right)\right]\times$$
$$\left(\mu_c\left(S_c+E_c+I_{cm}+I_{cs}+R_c\right)+\delta_{cm}I_{cm}+\delta_{cs}I_{cs}-\mu_c\left(S_c+E_c+I_{cm}+I_{cs}+R_c\right)-\delta_{cm}I_{cm}-\delta_{cs}I_{cs}\right)$$
$$+\left[\left(S_a-S_a^*\right)+\left(E_a-E_a^*\right)+\left(I_{am}-I_{am}^*\right)+\left(I_{as}-I_{as}^*\right)+\left(R_a-R_a^*\right)\right]\times \quad (63)$$
$$\left(\mu_a\left(S_a+E_a+I_{am}+I_{as}+R_a\right)+\delta_{am}I_{am}+\delta_{as}I_{as}-\mu_a\left(S_a+E_a+I_{am}+I_{as}+R_a\right)-\delta_{am}I_{am}-\delta_{as}I_{as}\right)$$

$$\frac{dv}{dt}=\left[\left(S_c-S_c^*\right)+\left(E_c-E_c^*\right)+\left(I_{cm}-I_{cm}^*\right)+\left(I_{cs}-I_{cs}^*\right)+\left(R_c-R_c^*\right)\right]$$
$$\times\left(-\mu_c\left(S_c-S_c^*\right)-\mu_c\left(E_c-E_c^*\right)-\mu_c\left(I_{cm}-I_{cm}^*\right)-\mu_c\left(I_{cs}-I_{cs}^*\right)-\mu_c\left(R_c-R_c^*\right)-\delta_{cm}\left(I_{cm}-I_{cm}^*\right)-\delta_{cs}\left(I_{cs}-I_{cs}^*\right)\right)$$
$$+\left[\left(S_a-S_a^*\right)+\left(E_a-E_a^*\right)+\left(I_{am}-I_{am}^*\right)+\left(I_{as}-I_{as}^*\right)+\left(R_a-R_a^*\right)\right] \quad (64)$$
$$\times\left(-\mu_a\left(S_a-S_a^*\right)-\mu_a\left(E_a-E_a^*\right)-\mu_a\left(I_{am}-I_{am}^*\right)-\mu_a\left(I_{as}-I_{as}^*\right)-\mu_a\left(R_a-R_a^*\right)-\delta_{am}\left(I_{am}-I_{am}^*\right)-\delta_{as}\left(I_{as}-I_{as}^*\right)\right)$$

This implies that

$$\frac{dv}{dt}=-\left[\left(S_c-S_c^*\right)+\left(E_c-E_c^*\right)+\left(I_{cm}-I_{cm}^*\right)+\left(I_{cs}-I_{cs}^*\right)+\left(R_c-R_c^*\right)\right]\times$$
$$\left[\begin{array}{l}\mu_c\left(S_c-S_c^*\right)+\mu_c\left(E_c-E_c^*\right)+\mu_c\left(I_{cm}-I_{cm}^*\right)+\mu_c\left(I_{cs}-I_{cs}^*\right)+\mu_c\left(R_c-R_c^*\right)+\delta_{cm}\left(I_{cm}-I_{cm}^*\right)+\\ \delta_{as}\left(I_{cs}-I_{cs}^*\right)-\left[\left(S_a-S_a^*\right)+\left(E_a-E_a^*\right)+\left(I_{am}-I_{am}^*\right)+\left(I_{as}-I_{as}^*\right)+\left(R_a-R_a^*\right)\right]\times\\ \mu_a\left(S_a-S_a^*\right)+\mu_a\left(E_a-E_a^*\right)+\mu_a\left(I_{am}-I_{am}^*\right)+\mu_a\left(I_{as}-I_{as}^*\right)+\mu_a\left(R_a-R_a^*\right)+\delta_{am}\left(I_{am}-I_{am}^*\right)\\ +\delta_{as}\left(I_{as}-I_{as}^*\right)\end{array}\right] \quad (65)$$

The derivative $\frac{dv}{dt}$ is negative and $\frac{dv}{dt}=0$ if and only if

$$S_c=S_c^*,E_c=E_c^*,I_{cm}=I_{cm}^*,I_{cs}=I_{cs}^*,R_c=R_c^* \quad (66)$$
$$S_a=S_a^*,E_a=E_a^*,I_{am}=I_{am}^*,I_{as}=I_{as}^*,R_a=R_a^*, \quad (67)$$

Also, every solution of system (3) with the starting condition moves nearer to ξ^* as $t\to\infty$.

This implies that the biggest compact invariant solution set in $\left\{\left(S_c,E_c,I_{cm},I_{cs},R_c,S_a,E_a,I_{am},I_{as},R_a\right)\in\Delta:\frac{dv}{dt}\le 0\right\}$ is a singleton set $\left\{\xi^*\right\}$. Thus, following Lasalle's invariant principle [12], the Legionnaires-endemic equilibrium is globally asymptotically stable in Δ with the condition that $R_E>1$.

3. Numerical Simulation

The impact of some important variables and parameters are hereby verified to determine the effectiveness of Legionnaires' diseases control strategies. Numerical analysis can provide insight on how best to tackle the transmission pattern of a disease outbreak and reduce death rate by studying the relative importance of underlying variables and parameters.

Table 2. The Legionnaire's Disease State Variables and Parameters

Symbols	value	References
π_C	500	Estimated
π_A	1000	Estimated
μ_C	0.002	[14]
μ_A	0.05	Estimated
α_C, α_A	0.1, 0.001	[15]
ϕ_C, ϕ_A	0.05, 0.20	[15]
χ_C, χ_A	0.0109,	[15]
δ_{CM}, δ_{CS}	0.33, 0.067	[21]
δ_{AM}, δ_{AS}	0.33, 0.03	[21]
σ_{CM}, σ_{CS}	0.0221, 0.3545	[15]
σ_{AM}, σ_{AS}	0.33, 0.34	[16]
γ	0.002	Estimated
q_C, q_A	0.5, 1	[16]
ω_C, ω_A	0.0952, 0.0241	[14]
η_C	0.05	Estimated
η	0.56	[15]
θ	0.3	Estimated
β	0.10	[15]

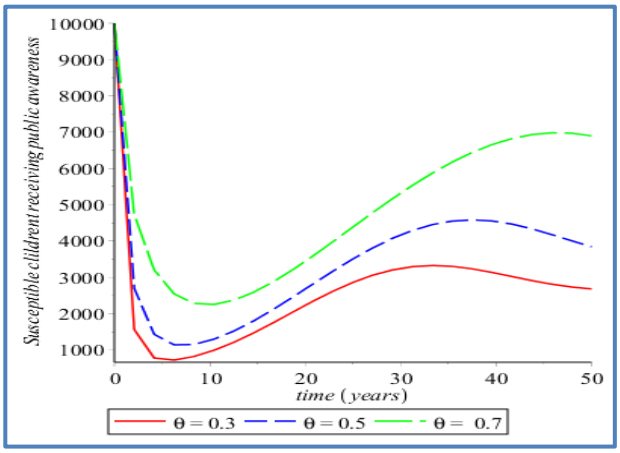
We adopt theoretical values that denotes the parameters, variables, and starting conditions to carry out respective simulations. The model equations were solved and graphical profiles obtained with parameters given in Table 2 using the Runge-Kutta built-in Maple 15 software.

Fig 2 show susceptible children population receiving public awareness. Fig 3 shows exposed children population receiving public awareness. Table 1 shows the parameter values applied in the simulation.

Fig 2 shows a diminishing trend in susceptible children population with varying public awareness. This happens for a specific period before experiencing a steady rise. This decline is because as more children population are educated the less they contract the disease. Whereas Fig 2 shows that the exposed children population rise sharply for a while before experiencing a gradual steady increase with varied public awareness. This happens because exposed children contracted the disease for a while before progressing out of the class.

Fig 4 exhibits the influence of public enlightenment on the susceptible adult population. Fig 5 depicts the influence of public enlightenment on the exposed adult population. Table 2 shows the parameter values applied in the simulation.

Fig 4 indicates a decreasing pattern in the susceptible adult population before exhibiting a gradual, steady increase with a varied public awareness rate. This happens because as susceptible adults are exposed, their population decreases for a while as they progress out of that class. Whereas Fig 5 shows that the exposed adult population grows rapidly to a certain value, then starts to reduce slowly whenever public awareness is given a positive boost. This growth is due to the influx of humans from the susceptible to the exposed classes.



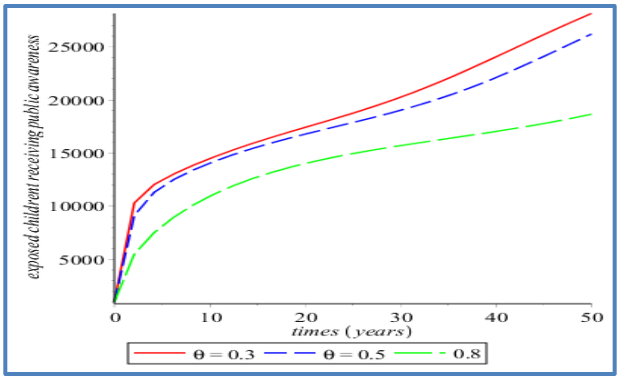
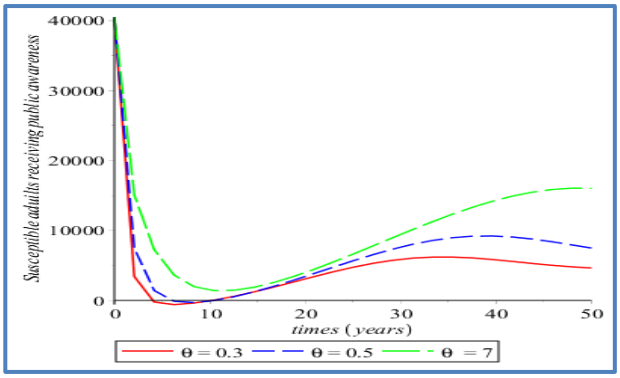


Fig 2 & 3. Effects of public awareness on children



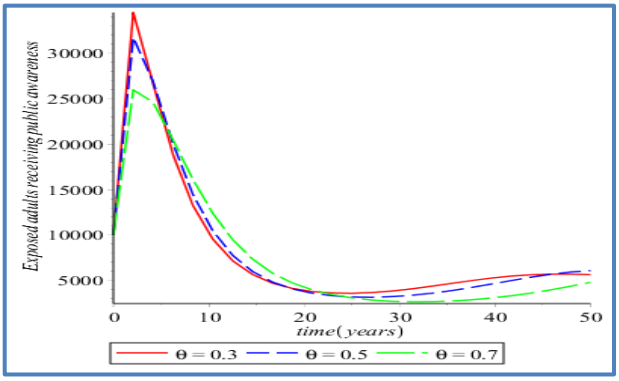


Fig 4 & 5: Effects of public awareness on adults

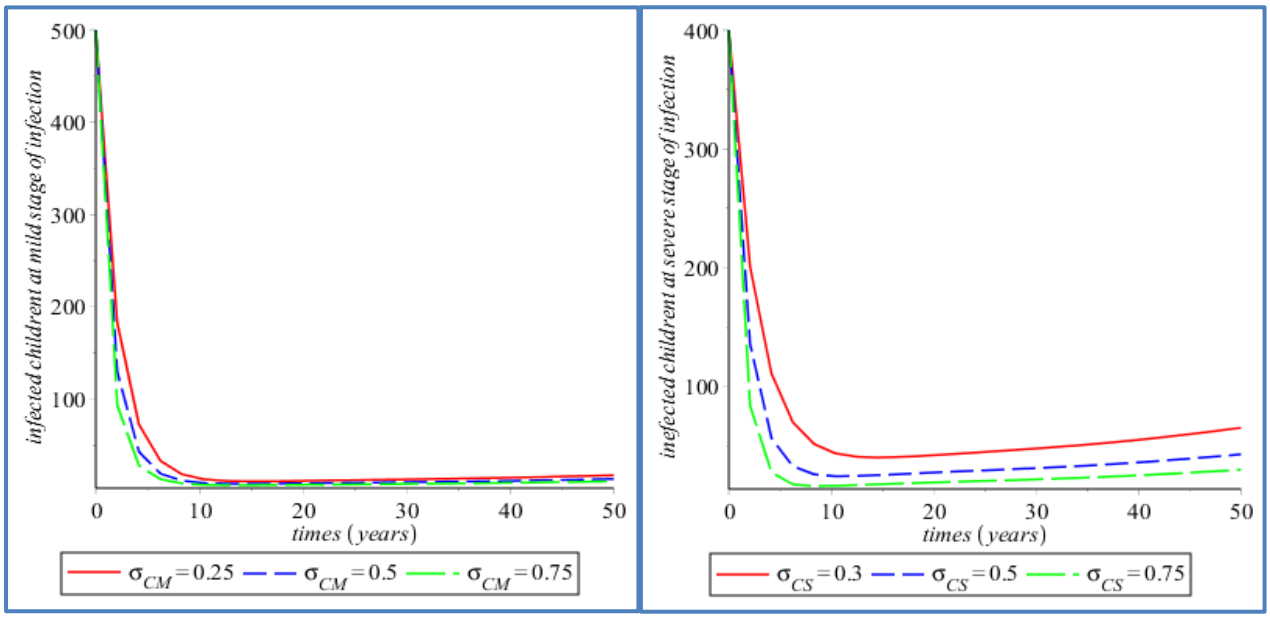


Fig 6 & 7. Effects of treatment on children

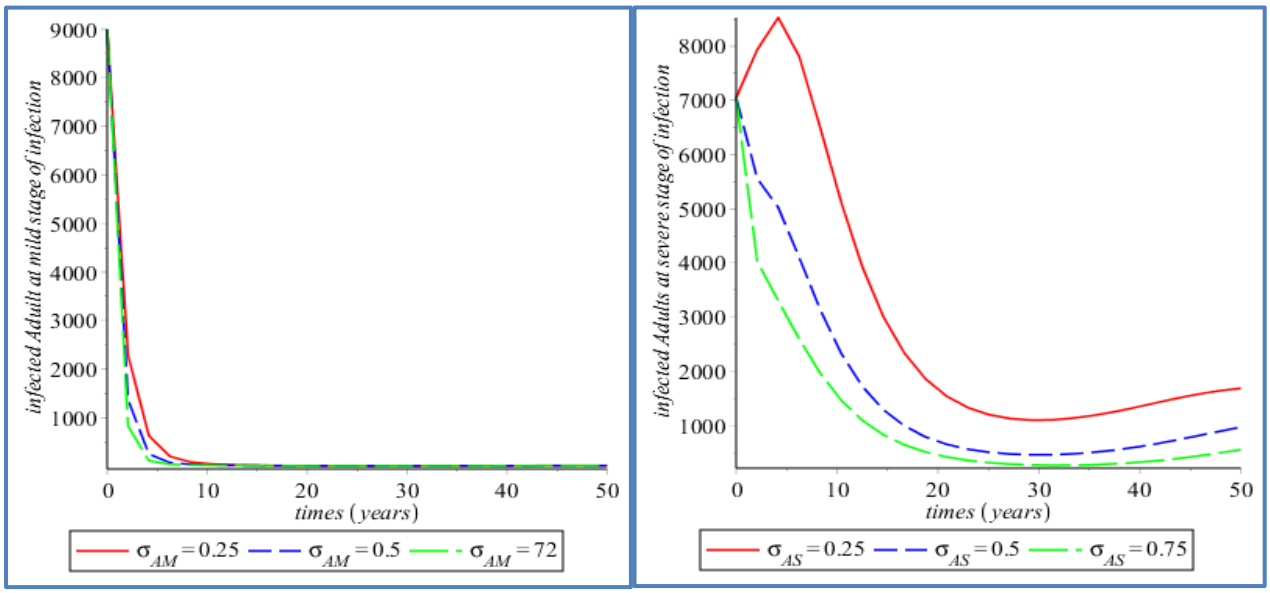


Fig 8 & 9. Effects of treatment on adults

Fig 6 shows the infected child population at a mild stage of infection. Fig 7 shows the infected child population at a severe stage of infection. Table 2 shows the parameter values applied in the simulation. Fig 6 and 7 show a decreasing pattern in the population of children suffering mild and severe infections.

Fig 8 shows an infected adult population at mild stage of infection. Fig 9 shows an infected adult population at severe stage of infection. Table 2 shows the parameter values applied in the simulation.

Fig 8 and 9 show a decreasing pattern in the adult population suffering mild and severe infections. This depicts how the population of adults suffering mild and severe infection responds to treatment and how the population decreases due to the progression into the recovered adult population.

4. Discussion

This study formulates a mathematical model to examine the behavior of Legionnaires' disease. The model is structured into 2 age groups (the child and adult populations) and comprises 10 subclasses using ordinary differential equations. The existence of Legionnaires-free equilibrium was established, and by using the control reproduction number, its stability was investigated globally for a threshold value below unity. This implies that it is necessary for deliberate efforts to be made by concerned agencies and stakeholders to ensure effective control strategies are sustained. The study further proved the existence of legionnaires-endemic equilibrium and investigated its stability, which was found to be globally stable if the actual (effective) reproduction number is greater than unity. This implies that unless adequate control measures

are sustained in society, Legionnaires' disease will persist in the environment.

5. Conclusion

The findings reveal that the prevalence of Legionnaires' disease can be drastically reduced in the human population whenever the actual (effective) reproduction number that depicts the spreading capacity is brought below unity. Globally, Legionnaires' disease changes from the mild to severe phase and thereafter changes from mild and severe to the recovered phase. The changes vary from one individual to another (the times that these progressions span ranges) from within a few weeks to some months [13-17]. In the analysis of the reproduction number, a high progression rate depicts a rapid movement from a particular phase of the disease manifestation to the next phase. These changes are seemingly determined by the environment and other related factors, such as in individuals already showing signs and symptoms of the disease. This study of Legionnaires' disease modeling, probably for the first time, has demonstrated the correspondence between the control measure (treatment) of infected individuals in the mild and severe phases of infection and the rates of change in the movement from the mild to the severe. Likewise, treatment patterns from the mild and severe stages to the recovered phase of Legionnaires' disease for controlling and preventing the disease threat are analyzed to understand the dynamics of Legionnaires' disease in the human population. However, with the aid of relevant stakeholders tasked with the responsibilities of intensifying public awareness campaigns on the risk of Legionnaires' disease in the human population, the disease burden can be reduced. Furthermore, administering effective treatments to humans exposed to Legionnaires' disease should be prioritized, as shown in the simulation results.

Declaration

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