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Dynamical Black holes in the FLRW universe

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ABSTRACT

Observations show that massive black holes, thought to be at the center of galaxies and galaxy clusters, are dynamical objects and interact with their environments. Black holes are characterized by their mass and angular momentum, and they are time-varying objects embedded in the cosmological Friedman-Lemaitre-Robertson-Walker (FLRW) universe. Finding analytical solutions for the dynamical black holes with all these properties is very important in theoretical physics. In the present work, we obtain Einstein's field equations for our predicted dynamical black hole geometry and obtain analytical solutions using the energy-momentum tensor obeying imperfect fluid dynamics. Misner-Sharp-Hernandez mass, turnaround radius, and apparent horizon are analyzed and compared with the recent observations.

1. Introduction

Scientific data show that the observable universe is expanding at an accelerating rate and is mostly filled with dark energy, dark matter, and baryonic matter [1, 2]. Dark energy, also known as vacuum energy, is thought to be responsible for this expansion and makes up 68% of the universe. The source of this unknown thing is one of the most important problem in theoretical and observational cosmology. In addition, when we observe the motions of galaxies, the rotational velocities away from the rotational axis have been measured to be higher than expected. This result predicts that there must be more matter in these galaxies than we observe. This structure is called dark matter and is estimated to make up 27% of the universe and has no interaction with electromagnetic fields and other types of matter. In this case, the amount of observed baryonic matter that interacts with the electromagnetic field is measured to be only about 5%.

There are many studies of possible candidates for dark energy and dark matter in the literature. For example, giant black holes believed to be existed at the centers of galaxies are predicted to be a source of dark energy [3]. In addition, some studies suggest that primordial black holes are candidates for dark matter [4, 5]. From an astrophysical point of view, black holes that are consistent with observations should be the ones that emit gravitational waves and interact with binary systems or galaxies in their nearby environment. For this reason, for example, the simplest Schwarzschild black hole geometry is an unrealistic and highly idealized black hole geometry because it is spherically symmetric, static, and defined in empty space-time [6, 7]. Reissner-Nordstrom black holes containing electric charge are also assumed in static and empty space-time geometry [8, 9]. Moreover, it is expected that charged black holes cannot form because the repulsion of the electric force would be greater than that of the gravitational force. For this reason, Reisner-Norstrom black holes are also considered to be mathematically consistent solutions. Kerr black holes are axisymmetric rotating black holes and are stationary, asymptotically flat and obtained for a static empty universe. Although Kerr black holes are a very important solution, it would be better to extend this geometry to the dynamical behavior of the universe [10].

In general relativity, it is very difficult to obtain analytical solutions of dynamical black holes. In particular, there are not many black hole solutions embedded in the FRLW spacetime geometry. The most important of these solutions is McVittie's metric [11], which has attracted much attention. For a recent discussions see reference [12]. These studies, which examine the effect of the cosmological universe on such localized systems, are important for understanding the evolution of black holes over time [13]. The non-rotating Thakurta solution [14] and the Sultana-Dyer black hole are other important studies [15–17]. However, the fact that the Sultana-Dyer solution includes negative energy solutions reduces its physical significance [18]. Dynamical black hole solutions and thermodynamic properties have been studied with alternative gravitational theories, and analytical solutions have been obtained [19, 20]. An eternal rotating black hole embedded in de Sitter spacetime and referred as Kerr-de-Sitter geometry presented in the works of [21, 22] are other important studies in the literature.

On scales larger than 250 million light-years, cosmological spacetime is found to be homogeneous and isotropic from observations of the cosmic background radiation [23]. For this reason, the universe is assumed to be filled with matter and energy expressed by the perfect fluid stress-energy tensor. However, recent redshift measurements show that the universe is not completely homogeneous and isotropic [24, 25]. The homogeneous and isotropic structure breaks down when examining galaxies and clusters smaller than the cosmological scale. Thus, it is more accurate to think of the structure of the universe as inhomogeneous. The definition of an imperfect energy-momentum tensor, which includes such dense objects, is also advantageous for obtaining analytical solutions of nonlinear

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field equations. Although stress energy tensor for the imperfect fluid were not widely accepted in the past, there are many valuable works in the literature [26–28]. A spherically symmetric geometry can be defined as the geometry of a body that is a source of gravitational attraction. The energy of these objects can be calculated at a distance using the mass energy balance. Their mass, defined as the Misner-Sharp-Hernandez (MSH) mass, can be associated with the strong gravitational forces of black holes [29]. This mass-energy balance is the fundamental basis of general relativity.

The radius at which the radial acceleration of particles far from the center of spherically symmetric gravitational massive objects in the FLRW universe becomes zero is known as the turnaround radius and is denoted by R_{tr}. At the limit of turnaround radius, the gravitational effect of the large spherical body and the dark energy repulsion of the cosmological background are balanced, and just outside the R_{tr}, the radial acceleration of the particles becomes zero. Thus, these particles escape from the the repulsion of gravity and move forever in the expanding universe. In this region, space and matter follow the Hubble flow of the cosmological expanding universe. Similarly, inside the R_{tr}, particles fall into the gravitational object. This radius is not considered an event horizon because particles can pass this radius if they are fast enough [30].

In the de Sitter cosmological universe, the turnaround radius of an object with mass M is $R_{tr} = \left(3GM/\Lambda c^2\right)^{1/3}$ and the matter density inside the turnaround radius is $\rho = M/V_{tr}$ where $V_{tr} = 4/3\pi R_{tr}^3$ volume inside the turnaround radius. This value is the lowest limit where there is no gravitational collapse of the structures. Structures with greater than this density within this radius experience gravitational collapse. For example, in the ΛCDM model, the turnaround radius R_{tr} and the size of the galaxies were obtained as compatible values [31, 32].

The content of this paper is as follows. In the next section we introduce the metric ansatz for the cosmological black hole and obtain the analytic solution of the field equations satisfying the energy-momentum tensor for the imperfect fluid for spherically symmetric space time. In the following section we present the metric in the different coordinates. In section 4, we calculate the turnaround radius, the apparent horizon and the MSH mass of the predicted spacetime, and we analyse the results and compare them with the literature. The paper ends with a conclusion in section 5.

2. The metric ansatz and field equations

Dynamical black hole solutions are important in terms of providing theoretical information about whether cosmic expansion will affect local systems or not. The black hole model embedded in FLRW cosmological spacetime and evolving in time can be simply described as follows,

$$ds^2 = -f(t,r)dt^2 + a(t)^2\left[\frac{1}{f(t,r)}dr^2 + r^2 d\Omega_2^2\right], \tag{1}$$

where the function, $f(t,r) = \left(1 - \frac{2M(t)}{r}\right)$. In this model, the Schwarzschild mass changes with time, and at sufficiently large radial distances we asymptotically obtain the FLRW cosmological flat-space geometry. Here $a(t)$ is the scale factor and gives information about the amount of expansion of spacetime. In this geometry, the Schwarzschild metric is obtained when $a(t)$ and $M(t)$ are constant, and the homogeneous and isotropic FLRW cosmological spacetime is obtained only when $M(t) = 0$ and also we get the Thakurta space-time for the constant $M(t) = M_0$. Since this geometry has no symmetry in the radial direction, it is not fulfilled by the energy-momentum tensor of a perfect fluid. The imperfect energy-momentum tensor can therefore be defined as follows [27, 28],

$$T^{ab} = (\rho + P_t)u^a u^b + P_t g^{ab} + (P_r - P_t)\, q^a q^b, \tag{2}$$

where P_r and P_t correspond to the radial and tangential pressures respectively. The anisotropy in fluid pressure of the source results from the tangential and the radial pressure difference $P_r - P_t$ [27, 28, 33]. u^a is the four-velocity vector satisfy the $u^a u_a = -1$ and q^a is the space like vector in radial direction and satisfied the $q^a q_a = 1$ (where $u^a = \left\{(-\left|g^{tt}\right|)^{1/2}, 0, 0, 0\right\}$ and $q^a = \left\{0, (g^{rr})^{1/2}, 0, 0\right\}$). In the homogeneous and isotropic model of the universe, there is no q^a vector, so the radial and angular discrepancy vanishes, and then the stress energy tensor is reduced to the structure that supports the matter as a perfect fluid.

The components of (2) become diagonal according to the geometry (1) and is given by,

$$T^0{}_0 = -\rho, \quad T^1{}_1 = P_r, \quad T^2{}_2 = T^3{}_3 = P_t. \tag{3}$$

Using Einstein field equations $G^a_{\ b} = 8\pi T^a_{\ b}$, the nonzero components satisfy the following equations,

$$\begin{aligned} G^0{}_0 &= -8\pi\rho = -\frac{3\dot{a}^2}{a^2 f} - \frac{2\dot{M}\dot{a}}{arf^2}, \\ G^1{}_1 &= 8\pi P_r = -\frac{\dot{a}^2}{a^2 f} - \frac{2\ddot{a}}{af} - \frac{2\dot{M}\dot{a}}{arf^2}, \\ G^2{}_2 &= G^3{}_3 = 8\pi P_t = -\frac{\dot{a}^2}{a^2 f} - \frac{2\ddot{a}}{af} - \frac{\ddot{M}}{rf^2} \\ &\qquad -\frac{5\dot{M}\dot{a}}{arf^2} - \frac{4\dot{M}^2}{r^2 f^3}, \\ G^0{}_1 &= -\frac{2\left(M\dot{a} + \dot{M}a\right)}{ar^2 f^2} = 0. \end{aligned} \tag{4}$$

All non-diagonal components of the stress-energy tensor $T^a_{\ b}$ vanish, in particular, the component $T^0_1 = 0$ implies that there is no radial mass flow and no accretion of cosmic fluid which can be fulfilled by the equation (4) to be zero,

$$\frac{\dot{a}}{a} + \frac{\dot{M}}{M} = 0 \tag{5}$$

which yields,

$$M(t) = \frac{M_0}{a(t)}, \tag{6}$$

where M_0 is a positive definite constant. Solving Einstein field equations, the energy density becomes

$$\rho = \frac{1}{8\pi f}\left(3H^2 - \frac{2M_0 H^2}{arf}\right), \tag{7}$$

the pressure in radial direction is

$$P_r = -\frac{1}{8\pi f}\left(3H^2 + 2\dot{H} - \frac{2M_0 H^2}{arf}\right), \tag{8}$$

and tangential pressure becomes

$$P_t = -\frac{1}{8\pi f^2}\left[3H^2 + 2\dot{H} - \frac{M_0}{ar}\left(10H^2 + 5\dot{H} - \frac{4M_0 H^2}{arf}\right)\right]. \tag{9}$$

The Ricci scalar is obtained as,

$$R = \frac{2}{f^2}\left[3\dot{H} + 6H^2 - \frac{M_0}{ar}\left(7\dot{H} + 18H^2 - \frac{4M_0 H^2}{arf}\right)\right].$$

(10)

The event horizon is a null hypersurface and defines the boundary of the black hole region of spacetime. From the Ricci scalar we obtain the event horizon from the singular points $f = 0$. The equation $f = 0$ yields $r = \frac{2M_0}{a}$, where $a(t)$ varies with time and hence event horizon $r(t)$ varies with time. In this sense we cannot localise the event horizon in dynamical geometries and it is more appropriate to find the apparent horizon. The apparent horizon is presented in section 4.

The scale factor $a(t)$ remains a free parameter. If we prefer to stay in the de Sitter cosmological universe, the scale factor $a(t) = a_0\, e^{\lambda t}$ satisfies all the field equations, which means that the universe expands exponentially.

3. Pseudo-Painleve-Gullstrand form

Studying in the Painleve coordinates become more useful especially when we work on the horizon of black holes. We define the areal radius as $R(t,r) = a(t)r$ in order to write equation (1) in Painleve-Gulstrand (PG) form, which we wrote in comoving coordinates in FLRW-Schwarzschild-like geometry [34] The differential of r becomes,

$$dr = \frac{1}{a}\left(dR - HRdt\right), \tag{11}$$

where $H(t) = \frac{\dot{a}}{a}$ is the Hubble parameter and overdot defines the derivative with respect to comoving time t. The PG form of line element becomes

$$ds^2 = -\left(1 - \frac{2M_0}{R} - \frac{H^2R^2}{1-\frac{2M_0}{R}}\right)dt^2 + \frac{dR^2}{1-\frac{2M_0}{R}} - \frac{2HR}{1-\frac{2M_0}{R}}dtdR + R^2 d\Omega_2^2. \tag{12}$$

Since this line element does not define the flat space for a constant time, the metric in (12) cannot be addressed as the PG coordinates but we can say it is the "pseudo-PG" type. To eliminate the cross term we introduce the time coordinate T as,

$$dT = \frac{1}{F(t,R)}\left(dt + \beta(t,R)dR\right). \tag{13}$$

For

$$\beta(t,R) = \frac{HR}{(1-\frac{2M_0}{R})^2 - H^2R^2}, \tag{14}$$

we get a metric similar to Schwarzshild-de Sitter-Kottler (S-dS-K) geometry as

$$ds^2 = -\left(1 - \frac{2M_0}{R} - \frac{H^2R^2}{1-\frac{2M_0}{R}}\right)dT^2 + \frac{dR^2}{\left(1 - \frac{2M_0}{R} - \frac{H^2R^2}{1-\frac{2M_0}{R}}\right)} + R^2 d\Omega_2^2. \tag{15}$$

Here, it should be noted that, the McVittie metric in S-dS-K coordinates is expressed as follows,

$$ds^2 = -\left(1 - \frac{2M_0}{R} - H^2R^2\right)dT^2 + \frac{dR^2}{(1-\frac{2M_0}{R} - H^2R^2)} + R^2 d\Omega_2^2. \tag{16}$$

If we compare our result with the McVittie metric, we have more general solution in (15), than the metric (16) [34].

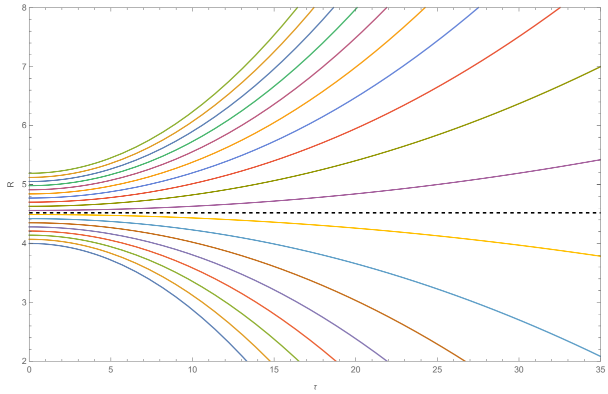

Fig. 1. Radial trajectories of massive test particles in equation (17) subject to zero initial velocity. The dashed black line is the location of the turnaround radius (R_{tr}). We used the values for the parameters $M = 1$ and $H = 0.1$.

Also in the [35], a metric similar to (15) (but not the same) was determined for the phantom matter, and the scale factor allows Big Rip solutions. In this sense, our work is a more general solution that leaves the scale factor as a free parameter and includes the expanding universe model.

4. Turnaround radius for dynamical black holes

The turnaround radius is obtained when the equations $d^2R/d\tau^2 = 0$ or $F = -dV_{eff}/d\tau = 0$ are satisfied. Here τ is the absolute time and R is the areal radius. Also in [36], the turnaround radius is obtained by equating the "local" part M_0 and the "cosmological" part $H^2R^3/2$ of the Hawking-Hayward/Misner Sharp-Hernandez mass to each other.

We can perform the radial geodesic equation for (12) and obtain the acceleration for a particle as,

$$\ddot{R}(\tau) = \left(\frac{R-3M_0}{(R-2M_0)^2}R^4 - \frac{M_0}{H^2}\right)\frac{H^2}{R^2} + R\dot{H}\dot{t}^2(\tau), \tag{17}$$

where an overdot denotes differentiation with respect to the proper time τ.

The position of the turnaround radius corresponds to $\ddot{R} = 0$ and therefore the acceleration of a unit mass vanishes. At this radius of the shell, the gravitational attraction of the central mass is balanced by the effect of the cosmological constant. For the scale factor $a(t) = exp(Ht)$, the $\dot{t}^2(\tau)$ term disappears for constant H and the radius of turnaround radius is obtained as follows,

$$R_1 = -2.66459 - 4.03589i\,,\ R_2 = -2.66459 + 4.03589i,$$
$$R_3 = 1.90397 - 0.396958i\,,\ R_4 = 1.90397 + 0.396958i,$$
$$R_5 = 4.52124. \tag{18}$$

where we have used the values for the parameters $M_0 = 1$ and $H = 0.1$ as in [36] for a simple comparison. Here only one of the roots of R has the real value and R_5 is denoted as the turnaround radius (R_{tr}), dashed black line in the figure:1). Here a test particle accelerates for $R > R_{tr}$ and moves away from the spherical mass M_0 (upper part of the figure: 1), and for $R < R_{tr}$ it will fall on the mass M_0 (lower part of the figure: 1), the radius $R = R_{tr}$ becomes an unstable equilibrium position.

Here, note that, for McVittie metric, the radial geodesic equation is presented as [36],

$$\ddot{R} = \left(R^3 - \frac{M_0}{H^2}\right)\frac{H^2}{R^2} + R\dot{H}\sqrt{1-\frac{2M_0}{r}}\dot{t}^2. \tag{19}$$

In this case, for example for the scale factor $a(t) = a_0 e^{Ht}$ (where H is constant), the second term vanishes and the turnaround radius becomes $R_c = (\frac{M_0}{H^2})^{1/3}$. This result is exactly the same as the Schwarzschild-de Sitter-Kottler spacetime, which is described as the spherically symmetric distribution of matter with mass M_0 in a spatially flat de Sitter background with a cosmological constant $\Lambda > 0$ (where the Hubble parameter is $H = \sqrt{\Lambda/3}$), [31].

In our work, from the equation (17), in the case of a constant H, we obtain the following equation for the turnaround radius,

$$\frac{R - 3M_0}{(R - 2M_0)^2} R^4 = R_c^3. \tag{20}$$

This equation is different from the McVittie case but becomes exactly the same for a sufficiently large radius R. In physical units, the equation (20) is represented as follows

$$\frac{R - \frac{3G_N M_0}{c^2}}{(R - \frac{2G_N M_0}{c^2})^2} R^4 = R_c^3. \tag{21}$$

Where $G_N = 6.674 \times 10^{-11} Nm^2/kg^2$ is the Newton's gravitational constant, $H = 67.3 \pm 1.1 (km/s)/Mpc = 2.18 \times 10^{-18} s^{-1}$ is the Hubble constant, $R_c = 11.2 \pm 0.1 Mpc \left(\frac{M_0}{10^{15} M_\odot}\right)^{1/3}$ is turnaround radius for the geometry S-deS-K space time and $c = 2.998 \times 10^8 m/s$ is the speed of light. For example for the Milky way mass approximately $M_0 = M_{MW} = 1.29 \times 10^{12} M_\odot$ [37, 38], substituting these values in to the (21) we get,

$$\begin{aligned} R_1 &= -5.96742 \times 10^{21} - 1.03359 \times 10^{22} i \\ R_2 &= -5.96742 \times 10^{21} + 1.03359 \times 10^{22} i, \\ R_3 &= 3.9 \times 10^{15} - 508489.i, \\ R_4 &= 3.9 \times 10^{15} + 508489.i, \\ R_5 &= 1.19348 \times 10^{22}. \end{aligned} \tag{22}$$

Except for R_5, all the roots are imaginary and have no physical meaning. The root of $R_5 = 1.19 \times 10^{22} m = 1.19 Mpc$ is our calculated value of the turnaround radius of the Milky Way. According to the latest observations, the estimated turnaround radius of the Milky Way is between the values $R_{tr} = 0.718 - 0.960 Mpc$ [39]. Our result is in agreement in degrees, but are slightly larger than the estimated value.

For dynamical black holes, the apparent horizon is a more useful concept and provides us a more stable calculation. On the other hand, the event horizon is not local in time and is not a practical way of looking at evolving black holes. The areal radius in (1) changes in time and is read as $R = a(t)r$ and the apparent horizon is obtained from the relation $\nabla^c R \nabla_c R = 0$ and satisfies,

$$1 - \frac{2M_0}{R} - \frac{H^2 R^2}{1 - \frac{2M_0}{R}} = 0. \tag{23}$$

If we make some simple mathematical work, we get the expression of the fourth order equation as $H^2 R^4 - R^2 + 4M_0 R - 4M_0^2 = 0$. This equation has four roots,

$$\begin{aligned} R_1 &= \frac{1 - \sqrt{1 - 8HM_0}}{2H}, \; R_2 = \frac{1 + \sqrt{1 - 8HM_0}}{2H}, \\ R_3 &= -\frac{1 + \sqrt{1 + 8HM_0}}{2H}, \; R_4 = -\frac{1 - \sqrt{1 + 8HM_0}}{2H} \end{aligned} \tag{24}$$

where R_3 is always negative and not physically accepted value. In our result R_2 reduces to areal radius for FLRW universe for $M_0 = 0$ in which $R_H = \frac{1}{H}$ and R_1 and R_4 disappears at this limit. In the static limit $H \to 0$, R_4 recovers the Schwarzschild event horizon, $R = 2M_0$ [40].

The Misner-Sharp-Hernandez mass which can be interpreted as a measure of mass inside a sphere of areal radius R at given time is M_{MHS} is defined as

$$1 - \frac{2M_{MSH}}{R} = \nabla^c R \nabla_c R, \tag{25}$$

and we get

$$M_{MSH} = M_0 + \frac{H^2 R^3}{2(1 - \frac{2M_0}{R})}. \tag{26}$$

This result has two terms, first one is the mass of local object M_0 and the second term on the right is the contribution of the cosmological background. Our result in (26) reduces to the Misner-Sharp mass of the Schwarzschild-de Sitter-Kottler spacetime and McVittie space for $M_0 \to 0$ and of the FLRW space for $M_0 = 0$. Note that, in (26), we have found a bigger mass value than the mass obtained in the McVittie space, in which $M_{MSH} = M_0 + \frac{H^2 R^3}{2}$ [41, 42]. (Here, in (26), for the limit $M_0 \to 0$, the second term in the denominator goes to zero faster than the first term on the right, and hence equation (26) becomes the S-deS-K mass). Also we recover the Schwarzschild mass for a constant scale factor in which $H = 0$.

5. Conclusion

In this work, we introduced a dynamical black hole embedded in the cosmological FLRW universe. Considering the cosmic fluid to be inhomogeneous, we have defined an imperfect fluid energy-momentum tensor and, using Einstein's general theory of relativity, we investigated the some of its properties. By obtaining the radial geodesic equation, we have found the turnaround radius, which is the radius of the spherical surface where the gravitational force acting on the object and the effect of dark energy are in equilibrium. We also obtained the apparent horizon and the MSH mass and concluded that our results are consistent with the literature. The McVittie metric in isotropic coordinates becomes S-deS-K spacetime when the Hubble parameter is constant, (where $a(t) = e^{Ht} = e^{\sqrt{\frac{\Lambda}{3}} t}$) and the turnaround radius becomes indistinguishable from that obtained in S-deS-K spacetime. However, in our work, as we expected, the turnaround radius was found to be slightly different and yields the S-deS-K result in the appropriate limit. In this respect, we can say that our predicted black hole metric can be a candidate for the dynamical black hole geometry and can define spacetime in a more general and detailed manner.

Decleration

Author Contribution: Conceive-DKÇ.; Design-ÖK.; Supervision-DKÇ; Computational Performance, Data Collection and/or Processing-DKÇ, ÖK; Analysis and/or Interpretation Literature Review-DKÇ, ÖK ; Writer-DKÇ, ÖK; Critical Reviews–DKÇ, ÖK

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