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A New Class of Operators: *LM*-operators

Erdal BAYRAM*[1]

[1] *Department of Mathematics, Tekirdağ Namık Kemal University, Tekirdağ, TÜRKİYE*

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ABSTRACT

This paper introduces a new class of operators called LM- operators, extending the concepts of L-weakly and M-weakly compact operators within the framework of Banach lattices. While L-weakly and M-weakly compact operators have established properties, LM-compact operators retain some of these characteristics but also diverge in certain respects. The study explores the relationships between LM-compact operators and some existing classes, and examines their algebraic properties and duality aspects.

1. Introduction

The classes of operators known as L-weakly and M-weakly compact were established by P. Meyer-Nieberg [1]. While these classes are subsets of weakly compact operators, they do not imply compactness; additionally, compact operators are not necessarily included in these categories.

While the analysis of weakly compact operators in a general Banach lattice is complex, the subclasses of L-weakly and M-weakly compact operators exhibit various order properties. There is a wealth of literature discussing the relationships between these subclasses, their connections to weakly compact operators, and other operator classes. Although a general operator class does not exist, it has been shown that the classes of regular L-weakly and regular M-weakly compact operators form a Banach lattice and possess numerous ordered properties (see [2]).

Recently, new operator classes related to L-weakly and M-weakly compact operators have been introduced, along with their associated properties. These include almost L- and M-weakly compact operators [3], pL- and pM-weakly compact operators [4], LW-compact operators [5], L-weakly and M-weakly demicompact operators [6], null almost L- and M-weakly compact operators [7], un L-weakly and un M-weakly compact operators [8], Generalizations of L- and M-weakly compact operators [9], limitedly L-weakly compact operators [10], statistical order compact operators [11]. These classes represent generalizations of L-weakly and M-weakly compact operators.

In this study, we propose a novel class of operators referred to as *LM-operators*, which are defined in relation to L-weakly and M-weakly compact operators. Our findings indicate that this new class retains some of the ordered properties of the original classes while lacking others. Additionally, we explore comparisons with compact and weakly compact operator classes, as well as investigate some of their algebraic properties and aspects of duality.

2. Materials and Methods

We use [12] and [13] as our primary sources concerning Banach lattices and operators on them. However, for the convenience of the reader, let us recall some definitions that this work involves.

In the remainder of this manuscript, we will consider E as a Banach lattice with norm dual E', B_E denoting the closed unit ball of E, and sol(A) representing the solid hull of the set A.

We refer to an operator as a linear and norm-bounded transformation. An operator T that maps from the Banach lattice E to the Banach lattice F is called positive if it satisfies $T(E^+) \subseteq F^+$. The set of all positive operators within the subclass $\mathcal{P}$ is denoted by $\mathcal{P}^+$. The notation TS indicates the composition of the operators T and S, while T^k refers to the operator T composed with itself k times, for $k \in \mathbb{N}^+$.

Definition 2.1. [13] Let E be a Banach lattice and X be a Banach space.

i. A non-empty bounded subset A of E is called L-weakly compact if every disjoint sequence contained in sol(A) converges to zero in norm.

ii. An operator T from X to E is defined as L-weakly compact if $T(B_X)$ is L-weakly compact set in E.

iii. An operator from E to X is defined as M-weakly compact if $\|Tx_n\| \to 0$ as $n \to \infty$ for every disjoint sequence (x_n) in B_E.

A Banach lattice E is said to have an order continuous norm if the condition $x_\alpha \downarrow 0$ in E leads to $\|x_\alpha\| \downarrow 0$. For example, all separable σ-Dedekind complete Banach lattices possess this characteristic. However, notable examples of Banach lattices that do not have order continuous norms include ℓ_∞ and c, the latter being the space of convergent sequences with the supremum norm. Moreover, the set $E^a = \{x \in E : \text{ every monotone sequence in } [0, |x|] \text{ converges}\}$ is the largest closed order ideal in E for which the induced norm is order

*Corresponding author: Department of Mathematics, Tekirdağ Namık Kemal University, Tekirdağ, TÜRKİYE E-mail address: **ebayram@nku.edu.tr**

continuous. For instance, for an atomless measure μ, we find that $(L^\infty(\mu))^a = \{0\}$ and $(\ell^\infty)^a = c_0$.

In our findings, we will use specific notations for a Banach space X and a Banach lattice E: $\mathcal{L}(X,E)$ represents the set of all linear and continuous operators, while $\mathcal{SC}(X,E)$, $\mathcal{W_L}(X,E)$, and $\mathcal{W_M}(E,X)$ denote the sets of all semi-compact operators, L-weakly compact operators, and M-weakly compact operators in $\mathcal{L}(X,E)$, respectively. Moreover, if $X = E$, we will use the shorthand notations $\mathcal{L}(E)$, $\mathcal{SC}(E)$, $\mathcal{W_L}(E)$, and $\mathcal{W_M}(E)$.

As shown in Propositions 2.4.10 and 3.6.2 of [13], E^a is indeed a closed order ideal that includes all L-weakly compact subsets. Consequently, $E^a = \{0\}$ (or $(E')^a = \{0\}$) if and only if $\mathcal{W}_L(E,F) = \{0\}$ (or $\mathcal{W}_M(E,F) = \{0\}$). Therefore, we assume that $E^a \neq \{0\}$ (or $(E')^a \neq \{0\}$).

3. Results

We have introduced a new operator class that builds upon the concepts of L-weakly and M-weakly compact operators, following a similar approach.

Definition 3.1. An operator $T : E \to E$ is classified as an LM-operator if $\lim \|Tx_n\| = 0$ for every disjoint sequence (x_n) that lies within sol$(T(B_E))$. The collection of all LM-operators is denoted by $\mathcal{LM}(E)$.

Remark 3.2. i) Every L-weakly and M-weakly compact operator qualifies as an LM-operator, which can be expressed as:

$$\mathcal{W_L}(E) \subseteq \mathcal{LM}(E) \quad \text{and} \quad \mathcal{W_M}(E) \subseteq \mathcal{LM}(E).$$

Indeed, if $T \in \mathcal{W_L}(E)$ and (x_n) is a disjoint sequence in sol$(T(B_E))$, then $\|x_n\| \to 0$. The continuity of T implies that $\|Tx_n\| \to 0$ as well. Similarly, if $T \in \mathcal{W_M}(E)$, then from the inclusion sol$(T(B_E)) \subseteq \|T\|B_E$, we also have $\|Tx_n\| \to 0$.

ii) However, an LM-operator is not necessarily weakly compact, meaning it does not have to be L-weakly or M-weakly compact. For instance, the operator

$$T : L^1[0,1] \to L^1[0,1], \quad Tf(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}] \\ f(x - \frac{1}{2}) & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$$

is not weakly compact, yet it is an LM-operator since $T^2 = 0$.

Conversely, a weakly compact operator is not necessarily an LM-operator. For example, the identity operator $Id : L^2[0,1] \to L^2[0,1]$ is weakly compact but does not qualify as an LM-operator since $\lim \|r_n\| \neq 0$ for the Rademacher sequence (r_n) in $B_{L^2[0,1]}$.

iii) The identity operator $Id : L^2[0,1] \to L^2[0,1]$ is neither weakly compact nor an LM-operator. Thus, in general, $\mathcal{LM}(E)$ is a proper subset of $\mathcal{L}(E)$.

iv) If E is an AL-space or an AM-space, then every weakly compact operator is an LM-operator, i.e., $\mathcal{W}(E) \subseteq \mathcal{LM}(E)$. This is because any weakly compact operator defined on an AL-space (or AM-space) is L-weakly compact (or M-weakly compact) and, consequently, qualifies as an LM-operator.

v) If the dual space E' possesses an order continuous norm, then every Dunford-Pettis operator is M-weakly compact. Therefore, if E is a Banach lattice with an order continuous dual norm, every Dunford-Pettis operator will also be an LM-operator.

Theorem 3.3. *$\mathcal{LM}(E)$ has the domination property.*

Proof. Suppose that $S, T \in \mathcal{L}(E)$ such that $0 \leq S \leq T$ and $T \in \mathcal{LM}(E)$. Then, the inclusion $sol\,(S\,(B_E)) \subseteq sol\,(T\,(B_E))$ holds. Thus, if we choose a disjoint sequence $(x_n) \subset sol\,(S\,(B_E))$ then $\|Tx_n\| \longrightarrow 0$ holds, so is $\|Sx_n\| \longrightarrow 0$. It means $S \in \mathcal{LM}(E)$. ☐

Lemma 3.4. *For any $T \in \mathcal{LM}(E)$ and $\varepsilon > 0$, there exists an element $u \in E^+$ such that $\left\|T\left(|Tx| - u\right)^+\right\| < \varepsilon$ for all $x \in B_E$.*

Proof. Assume that $A = sol\,(T\,(B_E))$. By Theorem 13.5 in [12], for $\varepsilon > 0$, there exists an element $u \in Id^+\,(A)$ such that $\left\|T\left(|z| - u\right)^+\right\| < \varepsilon$ for all $z \in A$. Hence $\left\|T\left(|Tx| - u\right)^+\right\| < \varepsilon$ for all $x \in B_E$. ☐

Proposition 3.5. *If $T \in \mathcal{LM}^+\,(E)$ then T^2 is a semi compact operator.*

Proof. As a consequence of the previous proposition, from the equality $(|Tx| - u)^+ = |Tx| - |Tx| \wedge u$, we see that

$$T^2\,(B_E) \subseteq T\,[-u,u] + \varepsilon B_E. \tag{1}$$

Since $Tu \geq 0$, it means $T^2 \in \mathcal{SC}\,(E)$. ☐

Remark 3.6. Contrary to Proposition 3.5, an operator $T \in \mathcal{L}\,(E)$ such that $T^2 \in \mathcal{SC}\,(E)$ does not need to be a LM-operator. For example, I_{ℓ_∞} identity operator of ℓ_∞ is not a LM-operator while $I^2_{\ell_\infty}$ is a semicompact operator.

Theorem 3.7. *For any Banach lattice E, the following assertions are equivalent.*

i. $T \in \mathcal{LM}(E)$.

ii. Every disjoint sequence (x_n) in $sol\,(T\,(B_E))$ converges uniformly to zero on $T'\,(B_{E'})$.

iii. Every disjoint sequence (f_n) in $sol\,(T'\,(B_{E'}))$ converges uniformly to zero on $T\,(B_E)$.

iv. $T' \in \mathcal{LM}\,(E')$.

Proof. $T \in \mathcal{LM}(E)$ if and only if for every disjoint sequence (x_n) in $sol\,(T\,(B_E))$, $\|Tx_n\| \to 0$, so $\sup\{|f\,(Tx_n)| : f \in B_{E'}\} = \sup\{|T'f\,(x_n)| : f \in B_{E'}\} \to 0$, i.e. every disjoint sequence (x_n) in $sol\,(T\,(B_E))$ converges uniformly to zero on $T'\,(B_{E'})$. By the Theorem 18.12 in [12], this fact is equivalent to that every disjoint sequence (f_n) in $sol\,(T'\,(B_{E'}))$ converges uniformly to zero on $T\,(B_E)$. This means that $T' \in \mathcal{LM}\,(E')$ from the definition of LM-operator. ☐

Theorem 3.8. *For any Banach lattice E, the following assertions are equivalent.*

i. $T \in \mathcal{LM}(E)$.

ii. $f_n\,(Tx_n) \to 0$ for every sequence (x_n) in B_E and for every disjoint sequence (f_n) in $sol\,(T'\,(B_{E'}))$.

Proof. $(i \Rightarrow ii)$ This is clear from previous theorem $(1 \Rightarrow 3)$ and the inequality

$$|f_n\,(Tx_n)| \leq \sup\{|f_n\,(Tx)| : x \in B_E\} \tag{2}$$

for every sequence (x_n) in B_E.

$(ii \Rightarrow i)$ Suppose that $\sup\{|f_n\,(Tx)| : x \in B_E\}$ does not converges to zero. Then, there exists some real $\varepsilon > 0$ and a subsequence $(f_{n_k}) \subset (f_n)$ such that

$\sup\{|f_{n_k}(Tx)| : f \in E'\} > \varepsilon$. Hence we can find a sequence (x_{n_k}) in B_E that satisfies $|f_{n_k}(Tx_{n_k})| > \varepsilon$, which contradicts to our assumption. □

Let W be a linear subset of the space $\mathcal{L}(E)$. If for every $T \in W$ and for every $S \in \mathcal{L}(E)$ the compositions ST and TS belong to W then W is called left ideal and right ideal, respectively, in $\mathcal{L}(E)$. For example, weakly compact operators, compact operators, Dunford-Pettis operators are left and right ideals in $\mathcal{L}(E)$ while semicompact regular operators, AM-compact regular operators are two sided ideals in the space of regular operators $L^r(E)$. However, L-weakly and M-weakly compact operators need not be two sided ideals in $\mathcal{L}(E)$ (see [14]). So, what can be said for regular operators?

In order to provide at least a partial answer to this question, let us first present proof of a well-known simple result below.

Lemma 3.9. *Let E be a Banach lattice, $T, S \in \mathcal{L}(E)$ and (x_n) be a sequence. If the inclusion $(x_n) \subset sol(TS(B_E))$ holds, then $(x_n) \subset sol(T(B_E))$.*

Proof. For every $n \in \mathbb{N}$, to being x_n in $sol(TS(B_E))$ implies that there exist $\alpha_n \in \mathbb{R}^+$ and $y_n \in B_E$ such that $|x_n| \leq \alpha_n |TS(y_n)|$. On the other hand since S is a bounded operator there exists $z_n \in B_E$ satisfying $y_n = \|S\| z_n$ for every $n \in \mathbb{N}$. Then,

$$|x_n| \leq \alpha_n |TS(y_n)| = \alpha_n |T(\|S\| z_n)| = \alpha_n \|S\| |T(z_n)| \quad (3)$$

holds which means $(x_n) \subset sol(T(B_E))$. □

The following result state that $\mathcal{LM}(E)$ is a algebraic left ideal in $\mathcal{L}(E)$.

Proposition 3.10. *If $T \in \mathcal{LM}(E)$ and $S \in \mathcal{L}(E)$ then $TS \in \mathcal{LM}(E)$.*

Proof. Let us choose a disjoint sequence (x_n) in $sol(TS(B_E))$. According to previous Lemma, $(x_n) \subset sol(T(B_E))$ holds. Since $T \in LM(E)$, $\|T(x_n)\| \to 0$, and, by the following inequality

$$\|TS(x_n)\| \leq \|S\| \|Tx_n\| \quad (4)$$

taking limit $n \to \infty$ we see $\|TS(x_n)\| \to 0$. □

Question 3.11. Is there any $S \in \mathcal{L}(E)$ such that $ST \notin \mathcal{LM}(E)$ where $T \in \mathcal{LM}(E)$?

By Proposition 3.10, if $T \in \mathcal{LM}(E)$ and the bounded operator S commutes with T then $ST \in \mathcal{LM}(E)$ holds. The commutant of an operator $T \in \mathcal{L}(X)$ is defined by the set $\{T\}' = \{S \in \mathcal{L}(X) : ST = TS\}$. Hence, if $S \in \{T\}'$ then $ST \in \mathcal{LM}(E)$.

Similarly, the super right commutant and super left commutant of an operator $S \in \mathcal{L}^+(F)$ are

$$[T\rangle = \{S \in \mathcal{L}^+(F) : TS \leq ST\}$$

and

$$\langle T] = \{S \in \mathcal{L}^+(F) : TS \geq ST\}$$

respectively. Clearly $[T\rangle \cap \langle T] = \{T\}'_+$ holds.

Corollary 3.12. *If $T \in \mathcal{LM}(E)$ and $S \in \langle T]$ then $ST \in \mathcal{LM}(E)$.*

Proof. From the Proposition 3.10, $ST \in \mathcal{LM}(E)$, so $TS \in \mathcal{LM}(E)$ since the domination property of $\mathcal{LM}(E)$. □

As a consequence of the Proposition 3.10, the following two results are easily seen.

Corollary 3.13. *For any Banach lattice E, the following assertions hold:*

i) If $T \in \mathcal{W}_{\mathcal{L}}(E)$ and $S \in \mathcal{L}(E)$ then $TS \in \mathcal{LM}(E)$.
ii) If $T \in \mathcal{W}_{\mathcal{M}}(E)$ and $S \in \mathcal{L}(E)$ then $ST \in \mathcal{LM}(E)$.

4. Conclusion

In this study, we have defined a new class of operators known as LM-operators, based on L-weakly and M-weakly compact operators. LM-operators serve as a generalization of both L-weakly and M-weakly compact operators while retaining some properties of these classes.

Furthermore, we have demonstrated that this new class preserves certain ordering properties, yet exhibits different characteristics in some aspects. In the literature, there are numerous studies on the properties of L-weakly and M-weakly compact operators, as well as their relationships with other operator classes. Our work contributes to the classification of operators by adding to the broad spectrum of operator classes presented in previous studies.

For future research, exploring LM-operators within the framework of specific properties of Banach lattices, as well as conducting a more in-depth examination of the relationships between this class and other operator classes, may offer new perspectives.

Decleration

Author Contribution: Conceive-E.B.; Design-E.B.; Supervision-E.B.; Computational Performance, Data Collection and/or Processing-E.B.; Analysis and/or Interpretation Literature Review-E.B.; Writer-E.B.; Critical Reviews–E.B.

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Orcid ID:

Erdal Bayram 0000-0001-8488-359X

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