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A Method of Fundamental Function for Fractional Linear Local Problems

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Research Article

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ABSTRACT

The literature of fundamental solutions is based on some fundamental functions such as Cauchy functions, Green functions, etc. Such functions and the approaches for their construction are valuable. The study aims to contribute the literature by introducing a method of such fundamental function for construction. Therefore, a linear ordinary differential equation with Caputo fractional derivative involving a coefficient in weighted Lebesgue space and locally a linear initial value condition is considered and studied by essentially using the concept of a special adjoint system of integral form. The solution of the adjoint system is the fundamental functional which enables the identification of the fundamental function for the problem.

1. Introduction

Linear differential problems with fractional derivative can be regarded as the generalizations of classical linear differential problems. Recently, such general problems are frequently studied in mathematics, physics, engineering, mechanics and many other disciplines [1–11]. The literature has plenty of papers related to the investigations and approaches on the fundamental solutions of such problems even if not as much as the classical ones. One of these approaches has been applied for the classical problems [12–17]. To the best of our knowledge, it has still not been applied for the fractional ones to determine the fractional Cauchy function or fractional Green function, which is the particular kind of the fundamental function corresponding to the problem. The aim at this dealing is to extend the approach to the fractional ones and hereby to contribute to the enrichment of literature.

This study aims to contribute to the growing literature on fractional differential equations by proposing a novel method for constructing fundamental functions specifically tailored to fractional linear local problems. The approach is grounded in the use of a special adjoint system and focuses on an equation with a Caputo fractional derivative and a coefficient belonging to a weighted Lebesgue space, subject to a linear initial condition. Through this framework, the concept of a fundamental functional is introduced, whose first component corresponds to the fundamental function of the problem. The proposed method distinguishes itself from existing techniques by leveraging the structural properties of solution spaces rather than relying on classical integration by parts.

Therefore, the rest of the study is organized as follows. In Section 2, some preliminaries are presented. In Section 3, the problem is stated in detail. In Section 4, the discussion on its solution is given. In Section 5, the illustration is carried out. In the final section, some results are emphasized.

2. Preliminaries

Let $\mathbb{R}$ be the set of real numbers. Let $G = (0, X)$ be a bounded open interval in $\mathbb{R}$. We say that $v : G \to \mathbb{R}$ is a weight function if it is Lebesgue measurable, a.e. a positive and locally integrable function on G. Let $L_{p,v}(G)$ with $1 \leq p < \infty$ be the space of Lebesgue measurable functions u on G such that

$$\|u\|_{L_{p,v}(G)} = \|u\|_{p,v} = (\int_0^X |u(x)|^p v(x)dx)^{\frac{1}{p}} < \infty.$$

Let $L_{\infty,v}(G)$ be the space of Lebesgue measurable and essentially bounded functions u on G such that

$$\|u\|_{L_{\infty,v}(G)} = \|u\|_\infty = ess \sup_{0<x<X} |u(x)|.$$

Theorem 2.1. *For $1 \leq p < \infty$ the space $L_{p,v}(G)$ and the space $L_{\infty,v}(G)$ are Banach spaces [18, 19].*

Theorem 2.2. *If $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for every linear bounded functional ℓ on the space $L_{p,v}(G)$ there is a unique $g \in L_{q,v}(G)$ such that*

$$\ell(f) = \int_0^X f(\xi)g(\xi)v(\xi)d\xi \quad for \quad all \quad f \in L_{p,v}(G).$$

Additionally $\|\ell\|_{L_{q,v}(G)} = \|g\|_{L_{q,v}(G)}$ [18, 19].

Let $\mathbb{N}$ denote the set of all natural numbers, $n \in \mathbb{N}$ and let $AC(G)$ and $AC^n(G)$ respectively denote the space of all absolutely continuous functions on G and the space of all real valued functions u which have continuous derivatives up to order $n-1$ on G such that $u^{(n-1)} \in AC(G)$. It is clear that $AC^1(G) = AC(G)$[3].

Furthermore, let us state several reminds the readers of the fractional derivative and integral [1–3, 5]:

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Definition 2.3. Let $u \in L_1(G)$. For almost all $x \in G$ and a positive α, the left and right Riemann-Liouville fractional integrals of order α are defined by

$$I^{\alpha}_{0+}u(x) \equiv \frac{1}{\Gamma(\alpha)}\int_0^x (x-\xi)^{\alpha-1}u(\xi)d\xi,$$

and

$$I^{\alpha}_{0-}u(x) \equiv \frac{1}{\Gamma(\alpha)}\int_x^X (\xi-x)^{\alpha-1}u(\xi)d\xi,$$

respectively, where Γ is the gamma function.

Definition 2.4. Let $u \in AC^n(G)$. For almost all $x \in G$ and a positive α, the left and right Riemann-Liouville fractional derivatives of order α are defined by

$$\begin{aligned} {}^{RL}D^{\alpha}_{0+}u(x) &\equiv \frac{d^n}{dx^n}(I^{n-\alpha}_{0+}u(x)) \\ &= \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n\int_0^x (x-\xi)^{n-\alpha-1}u(\xi)d\xi, \end{aligned}$$

and

$$\begin{aligned} {}^{RL}D^{\alpha}_{0-}u(x) &\equiv \frac{d^n}{dx^n}(I^{n-\alpha}_{0-}u(x)) \\ &= \frac{1}{\Gamma(n-\alpha)}(-\frac{d}{dx})^n\int_x^X (\xi-x)^{n-\alpha-1}u(\xi)d\xi, \end{aligned}$$

respectively, where $n \in \mathbb{N}$ and $n-1 < \alpha \le n$.

Definition 2.5. Let $u \in AC^n(G)$. For almost all $x \in G$ and a positive α, the left and right Caputo fractional derivatives of order α are defined by

$$\begin{aligned} {}^{C}D^{\alpha}_{0+}u(x) &\equiv I^{n-\alpha}_{0+}\frac{d^n}{dx^n}u(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-\xi)^{n-\alpha-1}u^{(n)}(\xi)d\xi, \end{aligned}$$

and

$$\begin{aligned} {}^{C}D^{\alpha}_{0-}u(x) &\equiv I^{n-\alpha}_{0-}(-\frac{d}{dx})^n u(x) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^X (\xi-x)^{n-\alpha-1}u^{(n)}(\xi)d\xi, \end{aligned}$$

respectively, where $n \in \mathbb{N}$ and $n-1 < \alpha \le n$.

Remark 2.6. For $\alpha \in \mathbb{N}$ the Riemann-Liouville and Caputo fractional derivatives of order α are the classical derivatives $\frac{d^n}{dx^n}$.

Theorem 2.7. *If $\alpha > 0$, $u \in C^{(n)}(G)$ and $n = [|\alpha|]+1$, then*

$${}^{C}D^{\alpha}_{0+}\ I^{\alpha}_{0+}u(x) = u(x), \quad {}^{C}D^{\alpha}_{0-}\ I^{\alpha}_{0-}u(x) = u(x),$$

and

$$I^{\alpha}_{0+}\ {}^{C}D^{\alpha}_{0+}\ u(x) = u(x) - u(0), I^{\alpha}_{0-}\ {}^{C}D^{\alpha}_{0-}\ u(x) = u(X) - u(x),$$
$$u'(0) = ... = u^{(n-1)}(0) = u'(X) = ... = u^{(n-1)}(X) = 0.$$

Definition 2.8. Let $1 \le p < \infty$ and $n = [|\alpha|]+1$. The space $W^{(\alpha)}_{p,v}(G)$ is defined by

$$W^{(\alpha)}_{p,v}(G) = \{u | u \in L_{p,v}(G) \cap AC^n(G), \quad {}^{C}D^{\alpha}_{0+}u \in L_{p,v}(G)\}$$

and a norm on the space is defined by

$$\|u\|_{W^{(\alpha)}_{p,v}(G)} = \|u\|_{p,v} + \|{}^{C}D^{\alpha}_{0+}u\|_{p,v} < \infty.$$

Lemma 2.9. *If $1 \le p < \infty$, then the space $W^{(\alpha)}_{p,v}(G)$ is a Banach space.*

3. Statement of The Problem

We consider the equation

$$(V_\alpha u)(x) \equiv {}^{C}D^{\alpha}_{0+}u(x) + A_0(x)u(x) = z_\alpha(x), \quad x \in G, \quad (3.1)$$

subject to classical local condition

$$V_0 u \equiv u(0) = z_0, \quad (3.2)$$

where $0 < \alpha < 1$, $A_0, z_\alpha \in L_{p,v}(G)$, $z_0 \in \mathbb{R}$ for $\frac{1}{p}+\frac{1}{q} = 1$.

We investigate an integral representation of the solution to problem (3.1)-(3.2) in the space $W^{(\alpha)}_{p,v}(G)$. Problem (3.1)-(3.2) is a linear problem which can be considered as an operator equation:

$$Vu = z, \quad (3.3)$$

with the linear operator $V = (V_\alpha, V_0)$ and $z = (z_\alpha(x), z_0)$. By the assumptions mentioned above we know that the operator $V = (V_\alpha, V_0) : W^{(\alpha)}_{p,v}(G) \to L_{p,v}(G) \times \mathbb{R}$ is bounded from $W^{(\alpha)}_{p,v}(G)$ into the Banach space $E_{p,v} \equiv L_{p,v}(G) \times \mathbb{R}$ of the elements $z = (z_\alpha(x), z_0)$ with

$$\|z\|_{E_{p,v}} = \|z_\alpha\|_{L_{p,v}(G)} + |z_0|, \quad 1 \le p < \infty.$$

If, for a given $z \in E_{p,v}$, problem (3.1)-(3.2) has a unique solution $u \in W^{(\alpha)}_{p,v}(G)$ such that $\|u\|_{W^{(\alpha)}_{p,v}(G)} \le c_0\|z\|_{E_{p,v}}$, then this problem is called a well-posed problem, where c_0 is a constant independent. Problem (3.1)-(3.2) is well-posed if and only if $V : W^{(\alpha)}_{p,v}(G) \to E_{p,v}$ is a (linear) homeomorphism.

4. Discussion

Problem (3.1)-(3.2) is analyzed by inspired by a novel concept of the adjoint problem [12–17]. This concept is introduced by the adjoint operator V^* of V. Any function $u \in W^{(\alpha)}_{p,v}(G)$ can be represented by [5]

$$u(x) = u(0) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-\xi)^{\alpha-1}\ {}^{C}D^{\alpha}_{0+}u(\xi)d\xi. \quad (4.1)$$

Then, we state that the operator V has an adjoint operator $V^* = (w_\alpha, w_0) : E_{q,v} \to E_{q,v}$ where $E_{q,v} \equiv L_{q,v}(G) \times \mathbb{R}$. By using the general form of a continuous linear functional on $E_{q,v}$ [5, 18, 19], we have

$$\begin{aligned} f(Vu) &\equiv \int_0^X f_\alpha(x)(V_\alpha u)(x)v(x)dx + f_0(V_0 u) \\ &= \int_0^X f_\alpha(x)v(x)\ [{}^{C}D^{\alpha}_{0+}u(x) + A_0(x)u(x)]dx \\ &+ f_0 u(0). \end{aligned}$$

By according to the representation (4.1), we obtain

$$\begin{aligned} & f(Vu) \equiv \int_0^X f_\alpha(x)v(x)[{}^{C}D^{\alpha}_{0+}u(x) \\ & + A_0(x)(u(0) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-\xi)^{\alpha-1}\ {}^{C}D^{\alpha}_{0+}\ u(\xi)d\xi)]dx \\ & + f_0 u(0) \\ & = \int_0^X (w_\alpha f)(\xi)\ {}^{C}D^{\alpha}_{0+}u(\xi)v(\xi)d\xi + (w_0 f)u(0) \\ & \equiv (V^* f)(u), \forall f = (f_\alpha(x), f_0) \in E_{q,v}, \forall u \in W^{(\alpha)}_{p,v}(G), \end{aligned} \quad (4.2)$$

where $f = (f_\alpha(x), f_0) \in E_{q,v}$ is considered as a linear bounded functional on $E_{p,v}$, $\frac{1}{p} + \frac{1}{q} = 1$ and

$$(w_\alpha f)(\xi) = f_\alpha(\xi) + \frac{1}{\Gamma(\alpha)} \int_\xi^X (s-\xi)^{\alpha-1} A_0(s) f_\alpha(s) ds$$
$$= f_\alpha(\xi) + I_{0-}^\alpha (A_0 f_\alpha)(\xi),$$
$$w_0 f = \int_0^X f_\alpha(x) A_0(x) v(x) dx + f_0.$$

The operator $w = (w_\alpha, w_0) : E_{q,v} \to E_{q,v}$ represented by $wf = (w_\alpha f, w_0 f)$ is linear and bounded by Minkowski and Hölder inequalities. The operator w is an adjoint operator for the operator V, in other words, $V^* = w$.

Now, Fredholm's alternative theorem can be stated in the context of solvability of the problem as follows:

Theorem 4.1. *If $1 < p < \infty$, then $Vu = 0$ has either only the trivial solution or a finite number of linearly independent solutions in $W_{p,v}^{(\alpha)}(G)$:*
(1) *If $Vu = 0$ has only the trivial solution in $W_{p,v}^{(\alpha)}(G)$, then also $wf = 0$ has only the trivial solution in $E_{q,v}$. Then, the operators $V : W_{p,v}^{(\alpha)}(G) \to E_{p,v}$ and $w : E_{q,v} \to E_{q,v}$ become linear homeomorphisms.*
(2) *If $Vu = 0$ has m linearly independent solutions $u_1, u_2, ..., u_m$ in $W_{p,v}^{(\alpha)}(G)$, then $wf = 0$ has also m linearly independent solutions*

$$f^{\star 1\star} = (f_\alpha^{\star 1\star}(x), f_0^{\star 1\star}), ..., f^{\star m\star} = (f_\alpha^{\star m\star}(x), f_0^{\star m\star})$$

in $E_{q,v}$. In this case, $Vu = z$ and $wf = \varphi$ have solutions $u \in W_{p,v}^{(\alpha)}(G)$ and $f \in E_{q,v}$ for given $z \in E_{p,v}$ and $\varphi \in E_{q,v}$ if and only if the conditions

$$\int_0^X f_\alpha^{\star i\star}(\xi) z_\alpha(\xi) v(\xi) d\xi + f_0^{\star i\star} z_0 = 0, \ i = 1, ..., m \tag{4.3}$$

and

$$\int_0^X \varphi_\alpha(\xi)\, {}^C D_{0+}^\alpha u_i(\xi) v(\xi) d\xi + \varphi_0 u_i(0) = 0, \ i = 1, ..., m \tag{4.4}$$

are satisfied, respectively.

Let us consider the following equation given in the form of a functional identity

$$(wf)(u) = u(x), \quad \forall u \in W_{p,v}^{(\alpha)}(G), \tag{4.5}$$

where $f = (f_\alpha(\xi), f_0) \in E_{q,v}$ is an unknown pair and x is a parameter in the closure $\overline{G}$ of G.

Definition 4.2. Let $f(x) = (f_\alpha(\xi, x), f_0(x)) \in E_{q,v}$ be a pair with parameter $x \in \overline{G}$. If $f = f(x)$ is a solution of (4.5) for a given $x \in \overline{G}$, then $f(x)$ is called a fundamental functional of V (or of (3.3)). The first component $f_\alpha(\xi, x)$ of the fundamental functional $f(x)$ corresponds to the fundamental function of V (or of (3.3)).

We can rewrite (4.5) as

$$\int_0^X (w_\alpha f)(\xi)\, {}^C D_{0+}^\alpha u(\xi) v(\xi) d\xi + (w_0 f) u(0)$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1}\, {}^C D_{0+}^\alpha u(\xi) d\xi + u(0),$$
$$\forall f \in E_{q,v}, \forall u \in W_{p,v}^{(\alpha)}(G). \tag{4.6}$$

Hence, we can obtain the following special adjoint system

$$(w_\alpha f)(\xi) = \frac{(x-\xi)^{\alpha-1} H(x-\xi)}{\Gamma(\alpha) v(\xi)}, \quad \xi \in G,$$
$$(w_0 f) = 1, \tag{4.7}$$

where $H(x-\xi)$ is Heaviside function on $\mathbb{R}$. (4.5) is equivalent to the system (4.7). Therefore, $f(x)$ is a fundamental functional if and only if $f(x)$ is a solution of the system (4.7) for an arbitrary $x \in \overline{G}$. For a solution $u \in W_{p,v}^{(\alpha)}(G)$ of (3.3) and a fundamental functional $f(x)$, we can rewrite (4.2) as

$$\int_0^X f_\alpha(\xi, x) z_\alpha(\xi) v(\xi) d\xi + f_0(x) z_0$$
$$= \int_0^X [\tfrac{(x-\xi)^{\alpha-1} H(x-\xi)}{\Gamma(\alpha) v(\xi)}]\, {}^C D_{0+}^\alpha u(\xi) v(\xi) d\xi + u(0). \tag{4.8}$$

Since the right hand side of (4.8) is $u(x)$, the following theorem is stated:

Theorem 4.3. *If (3.3) has at least one fundamental functional $f(x)$, then any solution $u \in W_{p,v}^{(\alpha)}(G)$ of (3.3) can be represented by*

$$u(x) = \int_0^X f_\alpha(\xi, x) z_\alpha(\xi) v(\xi) d\xi + f_0(x) z_0 \tag{4.9}$$

and also, $Vu = 0$ has only the trivial solution.

Theorem 4.4. *If there exists a fundamental functional, then it is unique. Also, a fundamental functional exists if and only if $Vu = 0$ has only the trivial solution.*

If $Vu = 0$ has a nontrivial solution, then a fundamental functional corresponding to $Vu = z$ does not exist. Then, $Vu = z$ usually has no solution unless z has a special structure. So, a representation of the existing solution of $Vu = z$ can be investigated in a generalized sense [12–17].

5. Example

Now, we present an example by considering $\alpha = \frac{1}{2}, X = 1, v(x) = 1, A_0(x) = z_\alpha(x) = 0 \in L_{p,v}(G), z_0 \in \mathbb{R}$. The corresponding special adjoint system (4.7) can be constructed in the following form

$$f_\alpha(\xi) = \frac{(x-\xi)^{\alpha-1} H(x-\xi)}{\Gamma(\alpha)}, f_0 = 1,$$

where $\xi \in G$. As can be seen, $f_\alpha(\xi)$ and f_0 have been directly obtained easily and quickly. Thus, the associated fundamental functional $f(x) = (f_\alpha(\xi, x), f_0(x))$ has been determined. The first component $f_\alpha(\xi, x) = f_\alpha(\xi)$ is the fundamental function and it can be written as follows

$$f_{\frac{1}{2}}(\xi) = \frac{(x-\xi)^{-\frac{1}{2}} H(x-\xi)}{\Gamma(\frac{1}{2})},$$

where $x, \xi \in (0, 1)$.

6. Conclusion

By the discussion, the proposed method of fundamental function essentially is different from the known methods for constructing the fundamental function. The structural properties of the space of solution are considered instead of integration by parts. We are of the opinion that it is valuable and important, thanks to its applicability easily to a very wide class of the linear ordinary differential equations with Caputo fractional derivative involving a coefficient in weighted Lebesgue space and locally a linear initial value condition. If the focal equation consists solely of the principal part (the leading term) with Caputo fractional derivative, then the fundamental function is established much more easily and quickly. The fundamental function corresponding to the considered problem is the first component of the fundamental functional corresponding to same problem. The considered study may be useful for the investigations on the existence and uniqueness of the solutions to the functional linear and nonlinear problems related to the focal problem.

Decleration

Author Contribution: Conceive-K.Ö.; Design-K.Ö.; Supervision-K.Ö.; Computational Performance, Data Collection and/or Processing-K.Ö.; Analysis and/or Interpretation Literature Review-K.Ö. ; Writer-K.Ö.; Critical Reviews–K.Ö. .

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